Internet Appendix for

*Loss Sharing in Central Clearinghouses: Winners and Losers**

Christian Kubitza
European Central Bank, Germany
christian.kubitza@ecb.europa.eu

Loriana Pelizzon
Leibniz Institute SAFE, Germany, Ca'Foscari University of Venice, Italy, and CEPR
pelizzon@safe-frankfurt.de

Mila Getmansky Sherman
Isenberg School of Management, University of Massachusetts Amherst, USA
msherman@isenberg.umass.edu

*The views expressed in this paper are the authors' and do not necessarily reflect those of the European Central Bank or the Eurosystem.

A Loss Sharing Rules in Practice

We investigate the Default Rules of LCH Limited Rates Service, one of the largest clearinghouses worldwide, as of September 2022 (available at https://www.lch.com/resources/rulebooks/lch-limited). Using the terminology of default rules (we report the relevant excerpts of the rule book below), a clearing member i's default fund contribution is approximately equal to

$$\begin{aligned}
\text{Contribution}_i &\approx \text{Non-Tolerance Contribution}_i && (45)\\
&= \text{Non-Tolerance Amount} \times \text{Non-Tolerance Weight}_i && (46)\\
&= \text{Service Fund Amount} \times \frac{\text{Uncovered Stress Loss}_i}{\sum_j \text{Uncovered Stress Loss}_j} && (47)\\
&\approx \text{Total Uncovered Stress Loss} \times \frac{\text{Stress Loss}_i - \text{Margin}_i}{\sum_j \text{Stress Loss}_j - \text{Margin}_j} && (48)\\
&\approx \sum VaR_i \times \frac{VaR_i}{\sum VaR_i} && (49)\\
&= VaR_i = -\bar{\sigma}_i \Phi^{-1}(\alpha_{stress}), && (50)
\end{aligned}$$

where, in the first step, we ignore an additional ("tolerance") contribution that is related to temporary forbearance of initial margin.[IA.1] In the final two steps, we assume that the stress testing approach (which determines stress losses) resembles a Value-at-Risk approach with confidence level α_{stress} and is additive (as in the case of a Normal distribution), in which case the contribution is equal to entity i's portfolio Value-at-Risk.

According to default rule 21 (b), loss sharing contributions are proportional to default fund contributions, which implies that entity i's allocated share of default losses equals

$$\frac{\bar{\sigma}_i \Phi^{-1}(\alpha_{stress})}{\sum_j (1 - D_j)\bar{\sigma}_j \Phi^{-1}(\alpha_{stress})} = \frac{\bar{\sigma}_i}{\sum_j (1 - D_j)\bar{\sigma}_j}, \tag{51}$$

which is equivalent to loss sharing based on net portfolio risk.

Finally, Swapclear's Default Fund Supplement rule S1 (a) implies that the default fund must be replenished within 30 days after default events.

In the following, we provide the relevant excerpts from the LCH Limited Default Rules (as of September 2022):

[IA.1] Rule SC2 (i) on page 113 states: *The "SwapClear Tolerance" which shall be the aggregate amount of temporary initial margin forbearance provided by the Clearing House to SwapClear Clearing Members to enable registration of SwapClear Contracts.*

From Schedule 6 Rates Service Default Fund Supplement - Part A Rates Service Default Fund Supplement - Swapclear S1, p.127 ff.:

> *(b) the "SwapClear Tolerance Weight" of an SCM [...] shall be calculated by dividing (x) the average SwapClear Tolerance Utilisation of the relevant SCM during the 20 business day period preceding the relevant SwapClear Determination Date [...] by (y) the total of such average SwapClear Tolerance Utilisations of all Non-Defaulting SCMs [...]*
>
> *(c) the value of the "SwapClear Tolerance Contribution Amount" of: (x) an SCM [...] shall be calculated by multiplying the SwapClear Tolerance Amount by the SCM's SwapClear Tolerance Weight [...]*
>
> *(d) the "SwapClear Non-Tolerance Amount" shall be the value of that portion of the Rates Service Fund Amount - SwapClear after deducting the SwapClear Tolerance Amount*
>
> *(e) the value of the "SwapClear Non-Tolerance Contribution Amount" for a given SCM [...] shall be calculated by multiplying the SwapClear Non-Tolerance Amount by the SCM's SwapClear Non-Tolerance Weight*
>
> *(f) the "SwapClear Non-Tolerance Weight" of an SCM shall be calculated by dividing (i) the Uncovered Stress Loss [...] by (ii) the total Uncovered Stress Loss [...]. An SCM's "Uncovered Stress Loss," [...] shall be determined by the Clearing House [...] by, inter alia, deducting the amount of eligible margin held by the Clearing House with respect to the relevant SwapClear Contracts [...] from the stress loss [...]*
>
> *(g) the "SwapClear Contribution" of: (x) an SCM [...] shall be the sum of (i) that SCM's SwapClear Non-Tolerance Contribution Amount [...] and (ii) that SCM's Tolerance Contribution Amount [...]*

From Schedule 6 Rates Service Default Fund Supplement CS2, p.112 ff.:

> *(b) "The "Non-Tolerance Amount" which shall be the sum of: (1) the Combined Loss Value - Limb (1); plus (2) an amount equal to 10 per cent of the Combined Loss Value - Limb (1)"*

From the general default rules 21 (b) (p.21):

the amount due by a Non-Defaulting Clearing Member in respect of an Excess Loss shall [...] be the Non-Defaulting Clearing Member's pro rata share of such loss arising upon the relevant Default calculated as the proportion of such Non-Defaulting Clearing Member's relevant Contribution [...] relative to the aggregate relevant Contributions [...] of all Clearing Members engaged in the Relevant Business other than the relevant Defaulter at the time of the relevant Default.

From Schedule 6 Rates Service Default Fund Supplement - Part A Rates Service Default Fund Supplement - Swapclear S1 (a), p.127:

[...] following a Default, any determinations on a SwapClear Determination Date and any such SwapClear Determination Date which might otherwise have occurred under this Rule S1 shall be suspended for the duration of the period (the "SwapClear Default Period") commencing on the date of such Default and terminating on the later to occur of the following dates:

(i) the date which is the close of business on the day falling 30 calendar days after the Rates Service Default Management Process Completion Date in relation to such Default [...]; and

(ii) where, prior to the end of the period referred to in sub-paragraph (i) above [...] one or more subsequent Defaults (each a "Relevant Default") occur, the date which is the close of business on the day falling 30 calendar days after the Rates Service Default Management Process Completion Date in relation to a Relevant Default which falls latest in time [...].

B Additional Results: Counterparty Risk Exposure

Corollary IA.1. *The larger derivatives' systematic risk exposure, the more beneficial is central clearing for counterparty risk exposure,* $\frac{\partial \Delta E_i}{\partial \beta} < 0$.

Central clearing reduces counterparty risk exposure if, and only if, $\eta_i < \bar{\eta}$, *i.e., if directionality is sufficiently low, with* $\bar{\eta} = \frac{f(K)-f(K-1)}{f(1)} \in (0,1)$. *The larger the number of derivative classes* K, *the lower is the portfolio directionality required for central clearing to reduce counterparty risk exposure,* $\frac{\partial \bar{\eta}}{\partial K} < 0$. *Figure IA.1 illustrates this result.*

Proof. Using Proposition 1 and Lemma IA.2, it is

$$\frac{\partial \Delta E_i}{\partial \beta} = \frac{\partial}{\partial \beta}\frac{f(K-1)}{f(K)} + \eta_i \frac{\partial}{\partial \beta}\frac{f(1)}{f(K)} < 0. \tag{52}$$

Moreover, it is

$$\Delta E_i < 0 \;\Leftrightarrow\; \eta_i < \frac{f(K)-f(K-1)}{f(1)}. \tag{53}$$

Hence, $\bar{\eta} = \frac{f(K)-f(K-1)}{f(1)}$. Since it is $\frac{f(K)-f(K-1)}{f(1)} = 1$ for $K = 1$ and $f(K) - f(K-1)$ is strictly decreasing with K (see Lemma IA.2), $\bar{\eta} < 1$ for all $K > 1$. The remaining result follows from

$$\frac{\partial \bar{\eta}}{\partial K} = \frac{\partial}{\partial K}\frac{f(K)-f(K-1)}{f(1)} < 0, \tag{54}$$

using Lemma IA.2. □

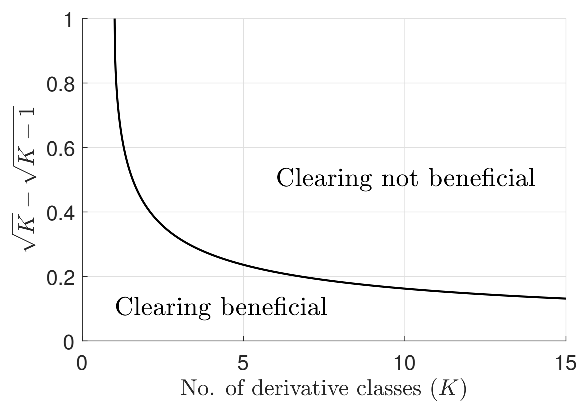

Figure IA.1. Maximum directionality for clearing to reduce counterparty risk exposure.
The figure depicts the function $\frac{f(K)-f(K-1)}{f(1)} = \sqrt{K} - \sqrt{K-1}$ for $\beta = 0$. If entity i's portfolio directionality η_i exceeds the function, central clearing does not reduce but increases counterparty risk exposure, i.e., is not beneficial. Instead, if η_i is below the function, central clearing reduces counterparty risk exposure, i.e., is beneficial.

C Additional Results: Cost of Collateral

In our baseline model, collateral protects counterparties against losses but we abstract from the cost of posting collateral. In this section, we extent the model by including a cost of collateral. Specifically, we denote by $c > 0$ the marginal cost of collateral. Thus, the collateral cost for entity i is cC_{ij}^{K} for uncleared positions with j and cC_{i}^{CCP} for cleared positions with the CCP. For consistency and without loss of generality, we assume that collateral costs arise only upon an entity's survival. Then, the impact of central clearing on expected default losses and collateral costs is given by

$$\Delta DLC_i = \frac{\mathbb{E}[(1-D_i)(DL_i^{K-1} + c\sum_{j\in\mathcal{N}_i} C_{ij}^{K-1} + cC_i^{CCP}) + LSC_i]}{\mathbb{E}[(1-D_i)DL_i^{K} + c\sum_{j\in\mathcal{N}_i} C_{ij}^{K}]}. \tag{55}$$

Whereas in the baseline model (with $c = 0$) a higher collateral requirement is unambiguously beneficial, with $c > 0$ it trades off with higher collateral costs, as we show in the following proposition.

Proposition IA.1 (Costly collateral)**.** *Assume that at least two entities have a portfolio that is not perfectly flat. Then,* ΔDLC_i *is equal to*

$$\Delta DLC_i = \frac{f(K-1)}{f(K)} + \frac{f(1)}{f(K)} \frac{\xi(\alpha_{CCP})\frac{w_i(\delta)}{G_i}\mathbb{E}[H] + c\eta_i\Phi^{-1}(\alpha_{CCP})}{\pi\xi(\alpha_{uc}) + c\Phi^{-1}(\alpha_{uc})} - 1, \tag{56}$$

where $H = \frac{\sum_{j=1,j\neq i}^{N} D_j G_j \eta_j}{w_i(\delta) + \sum_{j=1,j\neq i}^{N}(1-D_j)w_j(\delta)}$.

(1) If entity i has a flat portfolio, $\eta_i = 0$*, then the impact of central clearing on expected default losses and collateral costs is decreasing with the CCP's margin requirement,* $\frac{\partial \Delta DLC_i}{\partial \alpha_{CCP}} < 0$.

(2) If entity i's portfolio is not flat, $\eta_i > 0$*, and* $\alpha_{CCP} > 0$*, there exists* $0 < \hat{c} < \infty$ *such that the impact of central clearing on expected default losses and collateral costs is decreasing with the CCP's margin requirement if, and only if, the marginal cost of collateral c is below* $\hat{c}$*,*

$$\frac{\partial \Delta DLC_i}{\partial \alpha_{CCP}} < 0 \;\Leftrightarrow\; c < \hat{c}. \tag{57}$$

Proof. Using Lemma 1, the collateral posted by entity i to the CCP is equal to

$$C_i^{CCP} = \bar{\sigma}_i \Phi^{-1}(\alpha_{CCP}) = \eta_i G_i f(1) \Phi^{-1}(\alpha_{CCP}). \tag{58}$$

The total collateral posted by entity i to its bilateral counterparties in uncleared derivative classes $1, ..., K$ is equal to

$$\sum_{j \in \mathcal{N}_i} C_{ij}^K = \sum_{j \in \mathcal{N}_i} |v_{ij}| f(K) \Phi^{-1}(\alpha_{uc}) = G_i f(K) \Phi^{-1}(\alpha_{uc}). \tag{59}$$

Then, ΔDLC_i is equal to

$$\Delta DLC_i = \frac{\mathbb{E}[(1-D_i)(DL_i^{K-1} + c\sum_{j \in \mathcal{N}_i} C_{ij}^{K-1} + cC_i^{CCP}) + LSC_i]}{\mathbb{E}[(1-D_i)(DL_i^K + c\sum_{j \in \mathcal{N}_i} C_{ij}^K)]} \tag{60}$$

$$= \frac{\mathbb{E}[(1-D_i)(DL_i^{K-1} + cG_i(f(K-1)\Phi^{-1}(\alpha_{uc}) + \eta_i f(1)\Phi^{-1}(\alpha_{CCP}))) + LSC_i]}{\mathbb{E}[(1-D_i)(DL_i^K + cG_i f(K)\Phi^{-1}(\alpha_{uc}))]}. \tag{61}$$

Using Propositions 2 and 4 and following the steps in previous proofs, the impact of central clearing on the expected default losses and collateral cost of entity i is then given by

$$\Delta DLC_i = \frac{(1-\pi)\left(\pi G_i \xi(\alpha_{uc}) f(K-1) + \xi(\alpha_{CCP}) w_i(\delta) \mathbb{E}\left[\frac{\sum_{j=1, j\neq i}^N D_j \tilde{\sigma}_j}{w_i(\delta) + \sum_{j=1, j\neq i}^N (1-D_j) w_j(\delta)}\right]\right)}{(1-\pi)\left[\pi G_i \xi(\alpha_{uc}) f(K) + cG_i f(K)\Phi^{-1}(\alpha_{uc})\right]} \tag{62}$$

$$+ \frac{(1-\pi)\left(cG_i(f(K-1)\Phi^{-1}(\alpha_{uc}) + \eta_i f(1)\Phi^{-1}(\alpha_{CCP}))\right)}{(1-\pi)\left[\pi G_i \xi(\alpha_{uc}) f(K) + cG_i f(K)\Phi^{-1}(\alpha_{uc})\right]} - 1 \tag{63}$$

$$= \frac{f(K-1)}{f(K)} + \frac{\xi(\alpha_{CCP})\frac{w_i(\delta)}{G_i}\mathbb{E}\left[\frac{\sum_{j=1, j\neq i}^N D_j G_j \eta_j f(1)}{w_i(\delta) + \sum_{j=1, j\neq i}^N (1-D_j) w_j(\delta)}\right] + c\eta_i f(1)\Phi^{-1}(\alpha_{CCP})}{\pi \xi(\alpha_{uc}) f(K) + cf(K)\Phi^{-1}(\alpha_{uc})} - 1 \tag{64}$$

$$= \frac{f(K-1)}{f(K)} + \frac{f(1)}{f(K)} \frac{\xi(\alpha_{CCP})\frac{w_i(\delta)}{G_i}\mathbb{E}[H] + c\eta_i \Phi^{-1}(\alpha_{CCP})}{\pi \xi(\alpha_{uc}) + c\Phi^{-1}(\alpha_{uc})} - 1, \tag{65}$$

where $H = \frac{\sum_{j=1, j\neq i}^N D_j G_j \eta_j}{w_i(\delta) + \sum_{j=1, j\neq i}^N (1-D_j) w_j(\delta)}$.

The derivative of ΔDLC_i with respect to α_{CCP} is equal to

$$\frac{\partial \Delta DLC_i}{\partial \alpha_{CCP}} = \frac{f(1)}{f(K)} \frac{\xi'(\alpha_{CCP})\frac{w_i(\delta)}{G_i}\mathbb{E}[H] + c\eta_i \frac{1}{\varphi(\Phi^{-1}(1-\alpha_{CCP}))}}{\pi \xi(\alpha_{uc}) + c\Phi^{-1}(\alpha_{uc})} \tag{66}$$

$$= \frac{f(1)}{f(K)} \frac{-\frac{1-\alpha_{CCP}}{\varphi(\Phi^{-1}(1-\alpha_{CCP}))}\frac{w_i(\delta)}{G_i}\mathbb{E}[H] + c\eta_i \frac{1}{\varphi(\Phi^{-1}(1-\alpha_{CCP}))}}{\pi \xi(\alpha_{uc}) + c\Phi^{-1}(\alpha_{uc})} \tag{67}$$

$$= \frac{f(1)}{f(K)} \frac{1}{\varphi(\Phi^{-1}(1-\alpha_{CCP}))(\pi \xi(\alpha_{uc}) + c\Phi^{-1}(\alpha_{uc}))} \left(c\eta_i - (1-\alpha_{CCP})\frac{w_i(\delta)}{G_i}\mathbb{E}[H]\right), \tag{68}$$

using Lemma IA.2 and that the inverse function rule and the properties of the Normal distribution imply that

$$\frac{\partial \Phi^{-1}(\alpha_{CCP})}{\partial \alpha_{CCP}} = \frac{1}{\Phi'(\Phi^{-1}(\alpha_{CCP}))} \tag{69}$$

$$= \frac{1}{\varphi(\Phi^{-1}(\alpha_{CCP}))} = \frac{1}{\varphi(-\Phi^{-1}(1-\alpha_{CCP}))} = \frac{1}{\varphi(\Phi^{-1}(1-\alpha_{CCP}))}. \tag{70}$$

By assumption, $\alpha_{CCP} \in [0.5, 1)$ and, using that at least two entities have a non-flat portfolio and $\pi > 0$, $\mathbb{E}[H] > 0$.

(1) Clearly, if $\eta_i = 0$, then $\frac{\partial \Delta DLC_i}{\partial \alpha_{CCP}} < 0$.

(2) If $\eta_i > 0$, then

$$\frac{\partial \Delta DLC_i}{\partial \alpha_{CCP}} < 0 \Leftrightarrow c < (1-\alpha_{CCP})\frac{w_i(\delta)}{G_i}\mathbb{E}[H] > 0. \tag{71}$$

□

For entities' with a flat portfolio ($\eta_i = 0$), there is no collateral requirement due to zero net portfolio risk. Instead, for entities with $\eta_i > 0$, a higher collateral requirement for cleared positions, α_{CCP}, increases the benefit of central clearing (i.e., reduces ΔDLC_i) only if c is small, as we show in Proposition IA.1. In this case, the beneficial impact of collateral on default risk dominates. If, instead, c is sufficiently large, the adverse impact on collateral costs undermines clearing benefits.

The effect of the marginal cost of collateral c on ΔDLC_i is not obvious ex ante because it affects both cleared and uncleared positions. The following proposition sheds light on the role of c in core-periphery networks when losses are shared based on net risk and collateral requirements are the same for cleared and uncleared positions.

Proposition IA.2 (Costly collateral in core-periphery networks)**.** *Consider a core-periphery network and loss sharing based on net risk. Assume that $\alpha_{uc} = \alpha_{CCP}$. Then, for any entity $i \in \{1, ..., N\}$, the impact of central clearing on expected default losses and collateral costs is decreasing with the marginal cost of collateral,*

$$\frac{\partial \Delta DLC_i}{\partial c} < 0. \tag{72}$$

Proof. Let $g \in \mathcal{N}_{per}$ and $\delta = 0$. Using Proposition 4, the proof of Proposition 8, and that $\eta_g = 1$, it is

$$\mathbb{E}[LSC_g] = (1-\pi)\xi(\alpha_{CCP})\bar{\sigma}_g \mathbb{E}\left[\frac{\sum_{j=1,j\neq g}^{N} D_j\bar{\sigma}_j}{\bar{\sigma}_g + \sum_{j=1,j\neq g}^{N}(1-D_j)\bar{\sigma}_j}\right] \tag{73}$$

$$= (1-\pi)\xi(\alpha_{CCP})\eta_g G_g f(1)\mathbb{E}\left[\frac{\sum_{j=1,j\neq g}^{N} D_j\eta_j G_j f(1)}{\eta_g G_g f(1) + \sum_{j=1,j\neq g}^{N}(1-D_j)\eta_j G_j f(1)}\right] \tag{74}$$

$$= (1-\pi)\xi(\alpha_{CCP})\eta_g G_g f(1)\mathbb{E}\left[\frac{\sum_{j=1,j\neq g}^{N} D_j\eta_j G_j}{\eta_g G_g + \sum_{j=1,j\neq g}^{N}(1-D_j)\eta_j G_j}\right] \tag{75}$$

$$= (1-\pi)\xi(\alpha_{CCP})\eta_g G_g f(1)\frac{1-\pi^{2N/3}-1+\pi}{1-\pi} \tag{76}$$

$$= G_g(1-\pi)\xi(\alpha_{CCP})f(1)\frac{\pi-\pi^{2N/3}}{1-\pi} \tag{77}$$

and, therefore,

$$\Delta DLC_g = \frac{\mathbb{E}[(1-D_g)(DL_g^{K-1} + c\sum_{j\in\mathcal{N}_g} C_{ij}^{K-1} + cC_g^{CCP}) + LSC_g]}{\mathbb{E}[(1-D_g)DL_g^K + c\sum_{j\in\mathcal{N}_g} C_{ij}^K]} - 1 \tag{78}$$

$$= \frac{(1-\pi)\left(\pi G_g\xi(\alpha_{uc})f(K-1) + G_g\xi(\alpha_{CCP})f(1)\frac{\pi-\pi^{2N/3}}{1-\pi}\right)}{(1-\pi)\left[\pi G_g\xi(\alpha_{uc})f(K) + cG_g f(K)\Phi^{-1}(\alpha_{uc})\right]} \tag{79}$$

$$+ \frac{(1-\pi)\left(cG_g(f(K-1)\Phi^{-1}(\alpha_{uc}) + f(1)\Phi^{-1}(\alpha_{CCP}))\right)}{(1-\pi)\left[\pi G_g\xi(\alpha_{uc})f(K) + cG_g f(K)\Phi^{-1}(\alpha_{uc})\right]} - 1 \tag{80}$$

$$= \frac{f(K-1)}{f(K)} + \frac{f(1)}{f(K)}\frac{\pi\xi(\alpha_{CCP})\frac{1-\pi^{2N/3-1}}{1-\pi} + c\Phi^{-1}(\alpha_{CCP})}{\pi\xi(\alpha_{uc}) + c\Phi^{-1}(\alpha_{uc})} - 1. \tag{81}$$

The derivative of ΔDLC_g with respect to c is equal to

$$\frac{\partial\Delta DLC_g}{\partial c} = \pi\frac{f(1)}{f(K)}\frac{\Phi^{-1}(\alpha_{CCP})\xi(\alpha_{uc}) - \Phi^{-1}(\alpha_{uc})\xi(\alpha_{CCP})\frac{1-\pi^{2N/3-1}}{1-\pi}}{(\pi\xi(\alpha_{uc}) + c\Phi^{-1}(\alpha_{uc}))^2}. \tag{82}$$

If $\alpha_{uc} = \alpha_{CCP}$, then $\frac{\partial\Delta DLC_g}{\partial c} < 0$ if, and only if,

$$1-\pi < 1-\pi^{2N/3-1} \tag{83}$$

$$\Leftrightarrow \pi > \pi^{2N/3-1}, \tag{84}$$

which holds since $2N/3-1 > 1 \Leftrightarrow N > 3$ and $\pi < 1$, which hold by assumption.

If $h \in \mathcal{N}_{core}$ and for $\lim\tilde{\delta} \searrow 0$, using Proposition 4 and (the notation from) the proof of Proposition 8 it is

$$\lim_{\tilde{\delta}\searrow 0}\tilde{\delta}H_h = \mathbb{P}(\mathcal{D}_{per})\lim_{\tilde{\delta}\to 0}A_1 + (1-\mathbb{P}(\mathcal{D}_{per}))\lim_{\tilde{\delta}\to 0}A_2 \tag{85}$$

$$= \pi^{2N/3}\frac{6G_{per}}{(N-3)+6G_{per}}\frac{1-\pi^{N/3}}{1-\pi} \tag{86}$$

and

$$\lim_{\tilde{\delta}\searrow 0}\mathbb{E}[LSC_h] = \lim_{\tilde{\delta}\searrow 0}(1-\pi)\xi(\alpha_{CCP})(\tilde{\delta}\bar{\Sigma}_h+\bar{\sigma}_h)\mathbb{E}\left[\frac{\sum_{j=1,j\neq h}^{N}D_j\bar{\sigma}_j}{\tilde{\delta}\bar{\Sigma}_h+\bar{\sigma}_h+\sum_{j=1,j\neq h}^{N}(1-D_j)(\tilde{\delta}\bar{\Sigma}_j+\bar{\sigma}_j)}\right]$$

$$= (1-\pi)\xi(\alpha_{CCP})\bar{\Sigma}_h\lim_{\tilde{\delta}\searrow 0}\tilde{\delta}H_h \tag{87}$$

$$= (1-\pi)\xi(\alpha_{CCP})G_hf(1)\pi^{2N/3}\frac{6G_{per}}{(N-3)+6G_{per}}\frac{1-\pi^{N/3}}{1-\pi}, \tag{88}$$

and, therefore, using that $\eta_h = 0$ and for $\lim_{\tilde{\delta}\searrow 0}$,

$$\Delta DLC_h = \frac{\mathbb{E}[(1-D_h)(DL_h^{K-1}+c\sum_{j\in\mathcal{N}_h}C_{ij}^{K-1}+cC_h^{CCP})+LSC_h]}{\mathbb{E}[(1-D_h)DL_h^K+c\sum_{j\in\mathcal{N}_h}C_{ij}^K]}-1 \tag{89}$$

$$= \frac{(1-\pi)\left(\pi G_h\xi(\alpha_{uc})f(K-1)+\xi(\alpha_{CCP})G_hf(1)\pi^{2N/3}\frac{6G_{per}}{(N-3)+6G_{per}}\frac{1-\pi^{N/3}}{1-\pi}\right)}{(1-\pi)\left[\pi G_h\xi(\alpha_{uc})f(K)+cG_hf(K)\Phi^{-1}(\alpha_{uc})\right]} \tag{90}$$

$$+\frac{(1-\pi)\left(cG_h(f(K-1)\Phi^{-1}(\alpha_{uc})+f(1)\eta_h\Phi^{-1}(\alpha_{CCP}))\right)}{(1-\pi)\left[\pi G_h\xi(\alpha_{uc})f(K)+cG_hf(K)\Phi^{-1}(\alpha_{uc})\right]}-1 \tag{91}$$

$$= \frac{f(K-1)}{f(K)}+\frac{f(1)}{f(K)}\frac{\xi(\alpha_{CCP})\pi^{2N/3}\frac{6G_{per}}{(N-3)+6G_{per}}\frac{1-\pi^{N/3}}{1-\pi}}{\xi(\alpha_{uc})\pi+c\Phi^{-1}(\alpha_{uc})}-1, \tag{92}$$

which is decreasing with c. □

In core-periphery networks, expected loss sharing contributions per unit of cleared risk $f(1)$ are smaller than expected uncleared default losses per unit of uncleared risk $f(K)$ (see Proposition 8). A larger marginal cost of collateral c amplifies this difference between cleared and uncleared positions and, thereby, increases relative clearing benefits. This effect is particularly pronounced for core entities, which do not post collateral to the CCP due to their flat portfolio. In this case, a larger marginal collateral cost increases only the cost of uncleared but not of cleared positions, amplifying clearing benefits.

D Additional Statements

In many proofs, we make extensive use of the following property of the Normal distribution: For $Y \sim \mathcal{N}(\mu, \sigma^2)$ the truncated expected value is given by $\mathbb{E}[Y \mid Y > 0] = \mu + \sigma\frac{\varphi(-\mu/\sigma)}{\Phi(\mu/\sigma)}$, and thus $\mathbb{E}[\max(Y, 0)] = \mathbb{E}[Y \mid Y > 0]\Phi(\mu/\sigma) = \mu\Phi(\mu/\sigma) + \sigma\varphi(-\mu/\sigma)$, where $\varphi(\cdot)$ and $\Phi(\cdot)$ denote the probability density function and the cumulative density function of the standard normal distribution, respectively. From this property, we derive the following lemma:

Lemma IA.1. *Let $Y \sim \mathcal{N}(0, \sigma^2)$ and $C = \sigma\Phi^{-1}(\alpha)$ with $\alpha \in (0, 1)$. Then,*

$$\mathbb{E}[\max(Y - C, 0)] = \sigma\xi(\alpha), \tag{93}$$

where $\xi(\alpha) = (1-\alpha)\Phi^{-1}(1-\alpha) + \varphi(\Phi^{-1}(\alpha))$ with $\xi(0.5) = \varphi(0)$, $\xi'(\alpha) < 0$, $0 < \xi(\alpha) < \varphi(0)$ for all $\alpha \in (0.5, 1)$, and $\xi(\alpha) \to 0$ for $\alpha \to 1$.

Proof.

$$\begin{aligned}
\mathbb{E}[\max(Y - C, 0)] &= (-C)\Phi((-C)/\sigma) + \sigma\varphi(C/\sigma) && (94)\\
&= (-\sigma\Phi^{-1}(\alpha))\Phi((-\sigma\Phi^{-1}(\alpha))/\sigma) + \sigma\varphi(\sigma\Phi^{-1}(\alpha)/\sigma) && (95)\\
&= \sigma\left[(-\Phi^{-1}(\alpha))\Phi\left(-\Phi^{-1}(\alpha)\right) + \varphi(\Phi^{-1}(\alpha))\right] && (96)\\
&= \sigma\left[(-\Phi^{-1}(\alpha))\Phi\left(\Phi^{-1}(1-\alpha)\right) + \varphi(\Phi^{-1}(\alpha))\right] && (97)\\
&= \sigma\xi(\alpha) && (98)
\end{aligned}$$

with $\xi(\alpha) = (1-\alpha)\Phi^{-1}(1-\alpha) + \varphi(\Phi^{-1}(\alpha))$, where we use that $-\Phi^{-1}(\alpha) = \Phi^{-1}(1-\alpha)$. If $\alpha = 0.5$, then it is $\xi(\alpha) = 0.5\Phi^{-1}(0.5) + \varphi(\Phi^{-1}(0.5)) = \varphi(0)$. Using that $\varphi'(x) = (-x)\varphi(x)$ and the inverse function rule, the first derivative of ξ is equal to

$$\begin{aligned}
\xi'(\alpha) &= (-1)\Phi^{-1}(1-\alpha) + (1-\alpha)\frac{(-1)}{\Phi'(\Phi^{-1}(1-\alpha))} + (-\Phi^{-1}(\alpha))\varphi(\Phi^{-1}(\alpha))\frac{1}{\Phi'(\Phi^{-1}(\alpha))}\\
&= (-1)\Phi^{-1}(1-\alpha) + (1-\alpha)\frac{(-1)}{\varphi(\Phi^{-1}(1-\alpha))} + (-\Phi^{-1}(\alpha)) && (99)\\
&= (-1)\Phi^{-1}(1-\alpha) - \frac{1-\alpha}{\varphi(\Phi^{-1}(1-\alpha))} + \Phi^{-1}(1-\alpha) = -\frac{1-\alpha}{\varphi(\Phi^{-1}(1-\alpha))} < 0. && (100)
\end{aligned}$$

Moreover, it is

$$\lim_{\alpha\to 1}(1-\alpha)\Phi^{-1}(1-\alpha) + \lim_{\alpha\to 1}\varphi(\underbrace{\Phi^{-1}(\alpha)}_{\to\infty}) \tag{101}$$

$$= \lim_{\alpha\to 1}\frac{1-\alpha}{1/\Phi^{-1}(1-\alpha)} + 0 \tag{102}$$

$$= \lim_{\alpha\to 1}\frac{-1}{(-1)\times(\Phi^{-1}(1-\alpha))^{-2}\times\frac{1}{\Phi'(\Phi^{-1}(1-\alpha))}\times(-1)} \tag{103}$$

$$= \lim_{\alpha\to 1}(-1)\times(\Phi^{-1}(1-\alpha))^{2}\times\varphi(\Phi^{-1}(1-\alpha)) \tag{104}$$

$$= \lim_{\alpha\to 1}(-1)\times\frac{(\Phi^{-1}(1-\alpha))^{2}}{\frac{1}{\varphi(\Phi^{-1}(1-\alpha))}} \tag{105}$$

$$= \lim_{\alpha\to 1}(-1)\times\frac{2\times\Phi^{-1}(1-\alpha)\times\frac{(-1)}{\varphi(\Phi^{-1}(1-\alpha))}}{(-1)\times(\varphi(\Phi^{-1}(1-\alpha)))^{-2}\times\varphi'(\Phi^{-1}(1-\alpha))\times\frac{-1}{\varphi(\Phi^{-1}(1-\alpha))}} \tag{106}$$

$$= \lim_{\alpha\to 1}(-1)\times\frac{2\times\Phi^{-1}(1-\alpha)\times\frac{(-1)}{\varphi(\Phi^{-1}(1-\alpha))}}{(-1)\times(\varphi(\Phi^{-1}(1-\alpha)))^{-2}\times(-\Phi^{-1}(1-\alpha))\times\varphi(\Phi^{-1}(1-\alpha))\times\frac{-1}{\varphi(\Phi^{-1}(1-\alpha))}} \tag{107}$$

$$= \lim_{\alpha\to 1}-\frac{2\times\Phi^{-1}(1-\alpha)\times(\varphi(\Phi^{-1}(1-\alpha)))^{2}}{\Phi^{-1}(1-\alpha)\times\varphi(\Phi^{-1}(1-\alpha))} = \lim_{\alpha\to 1}(-2)\times\varphi(\Phi^{-1}(1-\alpha)) = 0, \tag{108}$$

using L'Hôpital's rule and the inverse function rule. Together with $\xi'(\alpha) < 0$, this implies $0 < \xi(\alpha) < \varphi(0)$ for all $\alpha \in (0.5, 1)$. From the above, it follows that $\xi(\alpha) \to 0$ for $\alpha \to 1$. □

Another result will be useful:

Lemma IA.2. *Define $f : (0,\infty) \to (0,\infty)$ by $f(K) = \sqrt{\beta^2\sigma_M^2K^2 + \sigma^2K}$ with $\sigma, \beta, \sigma_M > 0$. Then, $f'(K) > 0$, $f''(K) < 0$, and for all $K > 1$ it is*

$$\frac{\partial}{\partial K}[f(K) - f(K-1)] < 0. \tag{109}$$

Moreover, it is $\frac{\partial f}{\partial \beta} = \frac{\beta\sigma_M^2K^2}{f(K)}$, and $\frac{\partial}{\partial \beta}\frac{f(K_1)}{f(K_2)} < 0$ for all K_1, K_2 with $0 < K_1 < K_2$ and $\beta > 0$.

Proof. Rewrite $f(K) = \sqrt{X(K)}$ with $X(K) = \beta^2\sigma_M^2K^2 + \sigma^2K$. It is $f'(K) = \frac{2\beta^2\sigma_M^2K+\sigma^2}{2\sqrt{X(K)}} > 0$ and

$$f''(K) = \frac{2\beta^2\sigma_M^2 2\sqrt{X(K)} - \frac{2\beta^2\sigma_M^2K+\sigma^2}{\sqrt{X(K)}}(2\beta^2\sigma_M^2K+\sigma^2)}{4X(K)}, \tag{110}$$

which is negative, if and only if,

$$4\beta^2\sigma_M^2 X - (2\beta^2\sigma_M^2 K + \sigma^2)(2\beta^2\sigma_M^2 K + \sigma^2) < 0 \tag{111}$$

$$\Leftrightarrow 2\beta^2\sigma_M^2(2X - K(2\beta^2\sigma_M^2 K + \sigma^2)) - \sigma^2(2\beta^2\sigma_M^2 K + \sigma^2) < 0 \tag{112}$$

$$\Leftrightarrow 4\beta^2\sigma_M^2(X - \underbrace{(\beta^2\sigma_M^2 K^2 + \sigma^2 K)}_{=X}) - \sigma^4 < 0 \tag{113}$$

$$\Leftrightarrow \; -\sigma^4 < 0, \tag{114}$$

which holds by the assumption that $\sigma > 0$. Thus, $f'(K) < f'(K-1)$ and, therefore, $\frac{\partial}{\partial K}[f(K) - f(K-1)] = f'(K) - f'(K-1) < 0$. The derivative with respect to β is straightforward to calculate. Because $f(K) > 0$ for all $K > 0$, for $K_1, K_2 > 0$ it is

$$\frac{\partial}{\partial \beta}\frac{f(K_1)}{f(K_2)} < 0 \;\Leftrightarrow\; \frac{\partial}{\partial \beta}\frac{X(K_1)}{X(K_2)} < 0, \tag{115}$$

which, if $\beta > 0$, is equivalent to

$$\frac{\partial}{\partial \beta}\frac{\beta^2\sigma_M^2 K_1^2 + \sigma^2 K_1}{\beta^2\sigma_M^2 K_2^2 + \sigma^2 K_2} < 0 \tag{116}$$

$$\Leftrightarrow 2\beta\sigma_M^2 K_1^2(\beta^2\sigma_M^2 K_2^2 + \sigma^2 K_2) - 2\beta\sigma_M^2 K_2^2(\beta^2\sigma_M^2 K_1^2 + \sigma^2 K_1) < 0 \tag{117}$$

$$\Leftrightarrow \sigma^2(K_1^2 K_2 - K_2^2 K_1) + \beta^2\sigma_M^2(K_2^2 K_1^2 - K_2^2 K_1^2) < 0 \tag{118}$$

$$\Leftrightarrow \sigma^2(K_1 - K_2) < 0 \;\Leftrightarrow\; K_1 < K2. \tag{119}$$

□

E Proofs for Section 3

Lemma 1 (Portfolio risk). *The standard deviation of entity i's portfolio in a given derivative class is given by*

$$\bar{\sigma}_i = G_i \eta_i \sqrt{\beta^2 \sigma_M^2 + \sigma^2}. \tag{120}$$

Proof. The standard deviation of the portfolio in derivative class k is given by

$$\bar{\sigma}_i = \sqrt{\text{var}\left(\sum_{j \in \mathcal{N}_i} X_{ij}^k\right)} = \sqrt{\text{var}\left((\beta M + \varepsilon^K) \sum_{j \in \mathcal{N}_i} v_{ij}^k\right)} = \sqrt{(\beta^2 \sigma_M^2 + \sigma^2)} \left|\sum_{j \in \mathcal{N}_i} v_{ij}^k\right| \tag{121}$$

$$= G_i \eta_i \sqrt{\beta^2 \sigma_M^2 + \sigma^2}. \tag{122}$$

□

Proposition 1 (Impact of central clearing on counterparty risk exposure). *The impact of central clearing on entity i's counterparty risk exposure is equal to*

$$\Delta E_i = \frac{f(K-1) + \eta_i f(1)}{f(K)} - 1, \tag{123}$$

where $f(K) = \sqrt{\beta^2 \sigma_M^2 K^2 + \sigma^2 K}$. *The larger the portfolio directionality* η_i, *the less beneficial is central clearing for counterparty risk exposure,* $\frac{\partial \Delta E_i}{\partial \eta_i} > 0$.

Proof. The impact of central clearing is equal to

$$\Delta E_i = \frac{G_i f(K-1) + G_i \eta_i f(1)}{G_i f(K)} - 1 = \frac{f(K-1) + \eta_i f(1)}{f(K)} - 1, \tag{124}$$

where $f(K) = \sqrt{\beta^2 \sigma_M^2 K^2 + \sigma^2 K}$. Clearly, ΔE_i increases with η_i. □

F Proofs for Section 4

Proposition 2. *The expected default loss of entity i's uncleared positions in derivative classes 1 to K is equal to*

$$\mathbb{E}[DL_i^K] = \pi G_i \xi(\alpha_{uc}) \sqrt{\beta^2 \sigma_M^2 K^2 + \sigma^2 K}. \tag{125}$$

Proof. Entity i's expected default loss of uncleared positions in classes 1 to K is given by

$$\mathbb{E}\left[DL_i^K\right] = \sum_{j\in\mathcal{N}_i} \mathbb{E}\left[D_j \max\left(\sum_{k=1}^{K} X_{ij}^k - C_{ji}^K, 0\right)\right] \tag{126}$$

$$= \pi \sum_{j\in\mathcal{N}_i} \mathbb{E}\left[\max\left(\sum_{k=1}^{K} v_{ij}(\beta M + \sigma\varepsilon^k) - C_{ji}^K, 0\right)\right] \tag{127}$$

$$= \pi \sum_{j\in\mathcal{N}_i} \sqrt{\beta^2\sigma_M^2 K^2 v_{ij}^2 + K\sigma^2 v_{ij}^2}\,\xi(\alpha_{uc}) \tag{128}$$

$$= \pi G_i \xi(\alpha_{uc})\sqrt{\beta^2\sigma_M^2 K^2 + \sigma^2 K}, \tag{129}$$

where we use that defaults D_j are distributed independently of profits X_{ij}^k, that

$$C_{ji}^K = VaR_{\alpha_{uc}}\left(\sum_{k=1}^{K} X_{ji}^k\right) \tag{130}$$

$$= -\sqrt{var\left(\sum_{k=1}^{K} X_{ji}^k\right)}\Phi^{-1}(1-\alpha_{uc}) \tag{131}$$

$$= \sqrt{var\left(-\sum_{k=1}^{K} X_{ij}^k\right)}\Phi^{-1}(\alpha_{uc}) \tag{132}$$

$$= \sqrt{var\left(\sum_{k=1}^{K} X_{ij}^k\right)}\Phi^{-1}(\alpha_{uc}), \tag{133}$$

and Lemma IA.1. □

Proposition 3 (Impact of central clearing on the aggregate default loss)**.** *The expected aggregate default loss with central clearing is equal to*

$$ADL = \pi \sum_{i=1}^{N} G_i \left(\xi(\alpha_{CCP})\eta_i f(1) + \xi(\alpha_{uc}) f(K-1)\right), \tag{134}$$

where $f(K) = \sqrt{\beta^2\sigma_M^2 K^2 + \sigma^2 K}$*. The impact of central clearing on the expected aggregate default loss is equal to*

$$\Delta ADL = \frac{ADL - \sum_{i=1}^{N} DL_i^K}{\sum_{i=1}^{N} DL_i^K} = \frac{\xi(\alpha_{CCP})}{\xi(\alpha_{uc})}\frac{f(1)}{f(K)}\eta_{agg} + \frac{f(K-1)}{f(K)} - 1, \tag{135}$$

where $\eta_{agg} = \frac{\sum_{i=1}^{N} \left| \sum_{j \in \mathcal{N}_i} v_{ij} \right|}{\sum_{i=1}^{N} G_i}$ *is the average net-to-gross ratio.* $\Delta ADL < 0$ *holds only if*

$$\eta_{agg} < \frac{\xi(\alpha_{uc})}{\xi(\alpha_{CCP})}. \tag{136}$$

Proof. The CCP's expected total default losses is given by

$$\mathbb{E}\left[DL^{CCP}\right] = \sum_{j=1}^{N} \mathbb{E}\left[D_j \max\left(\sum_{g \in \mathcal{N}_j} X_{gj}^K - C_j^{CCP}, 0\right)\right] \tag{137}$$

$$= \pi \sum_{j=1}^{N} \mathbb{E}\left[\max\left(\sum_{g \in \mathcal{N}_j} v_{gj}^K (\beta M + \sigma \varepsilon^K) - C_j^{CCP}, 0\right)\right] \tag{138}$$

$$= \pi \sum_{j=1}^{N} \sqrt{var\left(\sum_{g \in \mathcal{N}_j} v_{jg}^K (\beta M + \sigma \varepsilon^K)\right)} \xi(\alpha_{CCP}) \tag{139}$$

$$= \pi \xi(\alpha_{CCP}) \sum_{j=1}^{N} \bar{\sigma}_j^K \tag{140}$$

$$= \pi \xi(\alpha_{CCP}) f(1) \sum_{j=1}^{N} G_j \eta_j, \tag{141}$$

with $f(K) = \sqrt{\beta^2 \sigma_M^2 K^2 + \sigma^2 K}$, where we use that

$$C_j^{CCP} = VaR_{\alpha_{CCP}}\left(\sum_{g=1}^{N} X_{jg}^K\right) \tag{142}$$

$$= -\sqrt{var\left(\sum_{g=1}^{N} X_{jg}^K\right)} \Phi^{-1}(1 - \alpha_{CCP}) \tag{143}$$

$$= \sqrt{var\left(-\sum_{k=1}^{K} X_{gj}^k\right)} \Phi^{-1}(\alpha_{CCP}) \tag{144}$$

$$= \sqrt{var\left(\sum_{k=1}^{K} X_{gj}^k\right)} \Phi^{-1}(\alpha_{CCP}), \tag{145}$$

and Lemma IA.1. Together with Proposition 2, the expected aggregate default loss with central clearing is thus equal to

$$\mathbb{E}\left[DL^{CCP} + \sum_{i=1}^{N} DL_i^{K-1}\right] \tag{146}$$

$$= \pi\xi(\alpha_{CCP}) f(1) \sum_{i=1}^{N} G_i \eta_i + \sum_{i=1}^{N} \pi G_i \xi(\alpha_{uc}) f(K-1) \tag{147}$$

$$= \pi \sum_{i=1}^{N} G_i \left(\xi(\alpha_{CCP})\eta_i f(1) + \xi(\alpha_{uc}) f(K-1)\right) \tag{148}$$

and without central clearing it is equal to

$$\mathbb{E}\left[\sum_{i=1}^{N} DL_i^{K}\right] = \pi\xi(\alpha_{uc}) \sum_{i=1}^{N} G_i f(K). \tag{149}$$

The derivation of ΔADL is straightforward. $\Delta ADL < 0$ is equivalent to

$$\frac{\xi(\alpha_{CCP})}{\xi(\alpha_{uc})} \frac{f(1)}{f(K)} \eta_{agg} + \frac{f(K-1)}{f(K)} < 1 \tag{150}$$

$$\Leftrightarrow \frac{\xi(\alpha_{CCP})}{\xi(\alpha_{uc})} \frac{f(1)}{f(K)} \eta_{agg} < 1 - \frac{f(K-1)}{f(K)} \tag{151}$$

$$\Leftrightarrow \eta_{agg} < \frac{\xi(\alpha_{uc})}{\xi(\alpha_{CCP}) f(1)} [f(K) - f(K-1)]. \tag{152}$$

The statement follows from

$$\frac{\xi(\alpha_{uc})}{\xi(\alpha_{CCP}) f(1)} [f(K) - f(K-1)] \leq \frac{\xi(\alpha_{uc})}{\xi(\alpha_{CCP}) f(1)} [f(1) - f(0)] = \frac{\xi(\alpha_{uc})}{\xi(\alpha_{CCP})}, \tag{153}$$

using that $f(K) - f(K-1)$ is strictly decreasing in K for all $K > 1$ (Lemma IA.2) and $f(0) = 0$. □

Corollary 1. *Central clearing reduces the expected aggregate default loss, $\Delta ADL < 0$, only if at least one of the following conditions holds:*

- $\alpha_{uc} < \alpha_{CCP}$
- $\eta_{agg} < 1$.

The latter condition is equivalent to $\min_{i \in \{1,\ldots,N\}} \eta_i < 1$.

Proof. From Lemma IA.1, $\alpha_{uc} \geq \alpha_{CCP}$ implies that $\xi(\alpha_{uc}) \leq \xi(\alpha_{CCP})$ and, thus, $\frac{\xi(\alpha_{uc})}{\xi(\alpha_{CCP})} \leq 1$. Together with Proposition 3 the first statement follows. For the second statement, note that the average net-to-gross ratio is a weighted average of individual entities' net-to-gross ratio,

$$\eta_{agg} = \frac{\sum_{i=1}^{N} G_i \eta_i}{\sum_{i=1}^{N} G_i}, \tag{154}$$

and, thus, $\eta_{agg} < 1$ requires that there exists at least one entity with $\eta_i < 1$. Vice versa, if there exists at least one entity j with $\eta_j < 1$, then $\eta_{agg} = \frac{G_j \eta_j + \sum_{i=1, i\neq j}^{N} G_i \eta_i}{\sum_{i=1}^{N} G_i} \leq \frac{G_i \eta_j + \sum_{i=1, i\neq j}^{N} G_i}{\sum_{i=1}^{N} G_i} < 1$. □

Proposition 4 (Expected loss sharing contribution and the impact of central clearing)**.** *With the loss sharing rule $w(\delta)$, clearing member i's expected loss sharing contribution is equal to*

$$\mathbb{E}[LSC_i(\delta)] = (1-\pi)\xi(\alpha_{CCP}) w_i(\delta) \mathbb{E}\left[\frac{\sum_{j=1, j\neq i}^{N} D_j \bar{\sigma}_j}{w_i(\delta) + \sum_{j=1, j\neq i}^{N} (1-D_j) w_j(\delta)}\right]. \tag{155}$$

The impact of central clearing on i's expected default loss is given by

$$\Delta DL_i = \frac{f(K-1)}{f(K)} + \frac{w_i(\delta) f(1)}{G_i f(K)} \frac{\xi(\alpha_{CCP})}{\xi(\alpha_{uc})} \frac{1}{\pi} \mathbb{E}\left[\frac{\sum_{j=1, j\neq i}^{N} D_j G_j \eta_j}{w_i(\delta) + \sum_{j=1, j\neq i}^{N} (1-D_j) w_j(\delta)}\right] - 1. \tag{156}$$

Proof. The expected loss sharing contribution of entity i with loss sharing rule $w(\delta)$ is given by

$$\begin{aligned}
\mathbb{E}[LSC_i] &= \mathbb{P}(D_i = 0) \mathbb{E}\left[\frac{w_i(\delta)}{\sum_{j=1}^{N} (1-D_j) w_j(\delta)} DL^{CCP} \mid D_i = 0\right] \\
&= \mathbb{P}(D_i = 0) \mathbb{E}\left[\frac{w_i(\delta)}{\sum_{j=1}^{N} (1-D_j) w_j(\delta)} \sum_{j=1}^{N} D_j \max\left(\sum_{g \in \mathcal{N}_g} X_{gj}^{K} - C_j^{CCP}, 0\right) \mid D_i = 0\right] \\
&= (1-\pi) \mathbb{E}\left[\mathbb{E}\left[\frac{w_i(\delta)}{\sum_{j=1}^{N} (1-D_j) w_j(\delta)} \sum_{j=1}^{N} D_j \xi(\alpha_{CCP}) \bar{\sigma}_j \mid D_1, ..., D_N\right] \mid D_i = 0\right] \\
&= (1-\pi) \xi(\alpha_{CCP}) w_i(\delta) \mathbb{E}\left[\frac{\sum_{j=1}^{N} D_j \bar{\sigma}_j}{\sum_{j=1}^{N} (1-D_j) w_j(\delta)} \mid D_i = 0\right] \\
&= (1-\pi) \xi(\alpha_{CCP}) w_i(\delta) \mathbb{E}\left[\frac{\sum_{j=1, j\neq i}^{N} D_j \bar{\sigma}_j}{w_i(\delta) + \sum_{j=1, j\neq i}^{N} (1-D_j) w_j(\delta)}\right],
\end{aligned}$$

using the definition of DL^{CCP} and the law of total expectation.

Using Proposition 2, the impact of central clearing for entity i is then given by

$$\Delta DL_i = \frac{(1-\pi)\pi G_i \xi(\alpha_{uc}) f(K-1) + (1-\pi)\xi(\alpha_{CCP}) w_i(\delta) \mathbb{E}\left[\frac{\sum_{j=1,j\neq i}^{N} D_j \bar{\sigma}_j}{w_i(\delta)+\sum_{j=1,j\neq i}^{N}(1-D_j)w_j(\delta)}\right]}{(1-\pi)\pi G_i \xi(\alpha_{uc}) f(K)} - 1$$
$$= \frac{f(K-1)}{f(K)} + \frac{w_i(\delta)}{G_i}\frac{\xi(\alpha_{CCP})}{\xi(\alpha_{uc})}\frac{f(1)}{\pi f(K)}\mathbb{E}\left[\frac{\sum_{j=1,j\neq i}^{N} D_j G_j \eta_j}{w_i(\delta)+\sum_{j=1,j\neq i}^{N}(1-D_j)w_j(\delta)}\right] - 1.$$

□

Corollary 2 (Aggregate loss sharing contributions)**.** *Conditional on at least one clearing member surviving, aggregate loss sharing contributions are equal to the CCP's total default loss.*

Unconditionally expected total loss sharing contributions are equal to the CCP's total expected default loss scaled by the survival probability of $N-1$ clearing members:

$$\mathbb{E}\left[\sum_{i=1}^{N} LSC_i(\delta)\right] = (1-\pi^{N-1})\mathbb{E}\left[DL^{CCP}\right]. \tag{157}$$

Proof. If $\sum_{i=1}^{N} D_i < N$, then

$$\sum_{i=1}^{N} LSC_i = \frac{\sum_{i=1}^{N}(1-D_i)w_i(\delta)}{\sum_{i=1}^{N}(1-D_i)w_i(\delta)} DL^{CCP} = DL^{CCP}. \tag{158}$$

Analogously to the analysis in the proof of Proposition 3, if all clearing members default, then the CCP's default loss is equal to

$$\mathbb{E}\left[DL^{CCP} \mid \sum_{i=1}^{N} D_i = N\right] = \mathbb{E}\left[\sum_{i=1}^{N} \max\left(\sum_{j\in\mathcal{N}_i} X_{ji}^{K} - C_i^{CCP}, 0\right)\right] = \sum_{i=1}^{N} G_i \eta_i \xi(\alpha_{CCP}) f(1). \tag{159}$$

Finally, by the law of total expectation, it holds that

$$\begin{aligned}\mathbb{E}[DL^{CCP}] &= \mathbb{P}\left(\sum_{i=1}^{N} D_i = N\right)\mathbb{E}\left[DL^{CCP} \mid \sum_{i=1}^{N} D_i = N\right] \\ &\quad + \mathbb{P}\left(\sum_{i=1}^{N} D_i < N\right)\mathbb{E}\left[DL^{CCP} \mid \sum_{i=1}^{N} D_i < N\right] \qquad (160) \\ \Leftrightarrow \mathbb{P}\left(\sum_{i=1}^{N} D_i < N\right)\mathbb{E}\left[DL^{CCP} \mid \sum_{i=1}^{N} D_i < N\right] &= \mathbb{E}[DL^{CCP}] \\ &\quad - \mathbb{P}\left(\sum_{i=1}^{N} D_i = N\right)\mathbb{E}\left[DL^{CCP} \mid \sum_{i=1}^{N} D_i = N\right]. \qquad (161)\end{aligned}$$

Hence, one can rewrite the expected aggregate loss sharing contributions as follows (using Proposition 3 and that $LSC_i = 0$ for all i if $\sum_{i=1}^{N} D_i = N$):

$$\begin{aligned}\mathbb{E}\left[\sum_{i=1}^{N} LSC_i\right] &= \mathbb{P}\left(\sum_{i=1}^{N} D_i < N\right)\mathbb{E}\left[\sum_{i=1}^{N} LSC_i \mid \sum_{i=1}^{N} D_i < N\right] && (162)\\ &= \mathbb{P}\left(\sum_{i=1}^{N} D_i < N\right)\mathbb{E}\left[DL^{CCP} \mid \sum_{i=1}^{N} D_i < N\right] && (163)\\ &= \mathbb{E}[DL^{CCP}] - \mathbb{P}\left(\sum_{i=1}^{N} D_i = N\right)\mathbb{E}\left[DL^{CCP} \mid \sum_{i=1}^{N} D_i = N\right] && (164)\\ &= \pi \sum_{i=1}^{N} G_i \eta_i \xi(\alpha_{CCP}) f(1) - \pi^N \sum_{i=1}^{N} G_i \eta_i \xi(\alpha_{CCP}) f(1) && (165)\\ &= (1-\pi^{N-1})\pi \sum_{i=1}^{N} G_i \eta_i \xi(\alpha_{CCP}) f(1) && (166)\\ &= (1-\pi^{N-1})\mathbb{E}\left[DL^{CCP}\right]. && (167)\end{aligned}$$

□

Proposition 5 (Loss sharing based on net risk)**.** *The impact of central clearing on the expected default loss of entity i is equal to*

$$\Delta DL_i = \frac{f(K-1)}{f(K)} + (\tilde{\delta} + \eta_i)\frac{\xi(\alpha_{CCP})}{\xi(\alpha_{uc})}\frac{f(1)}{f(K)}\frac{1}{\pi}\mathbb{E}\left[\frac{\sum_{j=1, j\neq i}^{N} D_j G_j \eta_j}{(\tilde{\delta}+\eta_i)G_i + \sum_{j=1,j\neq i}^{N}(1-D_j)(\tilde{\delta}+\eta_j)G_j}\right] - 1, \qquad (168)$$

where $f(K) = \sqrt{\beta^2\sigma_M^2 K^2 + \sigma^2 K}$*.* ΔDL_i *is*

(a) *decreasing with the collateral requirement for cleared contracts,* $\frac{\partial \Delta DL_i}{\partial \alpha_{CCP}} < 0$*, and increasing with the collateral requirement for uncleared contracts,* $\frac{\partial \Delta DL_i}{\partial \alpha_{uc}} > 0$*,*

(b) *increasing with the number of derivative classes,* $\frac{\partial \Delta DL_i}{\partial K} > 0$*, if, and only if,* $\alpha_{CCP} > c$*, where* $c > 0$ *is a constant,*

(c) *decreasing with the systematic risk exposure,* $\frac{\partial \Delta DL_i}{\partial \beta} < 0$*.*

Proof. Using Propositions 2 and 4, the impact of central clearing for entity i is given by

$$\begin{aligned}
\Delta DL_i &= \frac{(1-\pi)\pi G_i \xi(\alpha_{uc}) f(K-1) + (1-\pi)\xi(\alpha_{CCP})(\tilde{\delta}\bar{\Sigma}_i + \bar{\sigma}_i)\mathbb{E}\left[\frac{\sum_{j=1,j\neq i}^N D_j \bar{\sigma}_j}{\tilde{\delta}\bar{\Sigma}_i + \bar{\sigma}_i + \sum_{j=1,j\neq i}^N (1-D_j)(\tilde{\delta}\bar{\Sigma}_j + \bar{\sigma}_j)}\right]}{(1-\pi)\pi G_i \xi(\alpha_{uc}) f(K)} - 1 \\
&= \frac{(1-\pi)\pi G_i \xi(\alpha_{uc}) f(K-1) + (1-\pi)\xi(\alpha_{CCP})(\tilde{\delta}+\eta_i) G_i f(1)\mathbb{E}\left[\frac{\sum_{j=1,j\neq i}^N D_j G_j \eta_j f(1)}{(\tilde{\delta}+\eta_i)G_i f(1) + \sum_{j=1,j\neq i}^N (1-D_j)(\tilde{\delta}+\eta_j) G_j f(1)}\right]}{(1-\pi)\pi G_i \xi(\alpha_{uc}) f(K)} - 1 \\
&= \frac{f(K-1)}{f(K)} + (\tilde{\delta}+\eta_i)\frac{\xi(\alpha_{CCP})}{\xi(\alpha_{uc})}\frac{f(1)}{f(K)}\frac{1}{\pi}\mathbb{E}\left[\frac{\sum_{j=1,j\neq i}^N D_j G_j \eta_j}{(\tilde{\delta}+\eta_i)G_i + \sum_{j=1,j\neq i}^N (1-D_j)(\tilde{\delta}+\eta_j)G_j}\right] - 1,
\end{aligned}$$

where $f(K) = \sqrt{\beta^2\sigma_M^2 K^2 + \sigma^2 K}$, using that D_i and D_j are independently distributed for $i \neq j$. Define

$$H = \frac{1}{\pi}\mathbb{E}\left[\frac{\sum_{j=1,j\neq i}^N D_j G_j \eta_j}{G_i(\tilde{\delta}+\eta_i) + \sum_{j=1,j\neq i}^N (1-D_j) G_j(\tilde{\delta}+\eta_j)}\right].$$

It is $H > 0$.

(a) The derivative of ΔDL_i with respect to α_{CCP} is equal to

$$\frac{\partial \Delta DL_i}{\partial \alpha_{CCP}} = \frac{\xi'(\alpha_{CCP})}{\xi(\alpha_{uc})}(\tilde{\delta}+\eta_i)\frac{f(1)}{f(K)} H < 0 \tag{169}$$

and the derivative with respect to α_{uc} is equal to

$$\frac{\partial \Delta DL_i}{\partial \alpha_{uc}} = -\frac{\xi'(\alpha_{uc})\xi(\alpha_{CCP})}{\xi(\alpha_{uc})^2}(\tilde{\delta}+\eta_i)\frac{f(1)}{f(K)} H > 0, \tag{170}$$

using in both cases that $\xi'(\alpha) < 0$ from Lemma IA.1.

(b) The derivative of ΔDL_i with respect to K is equal to

$$\frac{\partial \Delta DL_i}{\partial K} = \frac{f'(K-1)f(K) - f'(K)f(K-1)}{f^2(K)} - f'(K)\frac{f(1)}{f^2(K)}(\tilde{\delta}+\eta_i)\frac{\xi(\alpha_{CCP})}{\xi(\alpha_{uc})}H \quad (171)$$

$$= \frac{f'(K-1)f(K) - f'(K)\left[f(K-1) + f(1)(\tilde{\delta}+\eta_i)\frac{\xi(\alpha_{CCP})}{\xi(\alpha_{uc})}H\right]}{f^2(K)}, \quad (172)$$

which is positive if, and only if,

$$f'(K-1)f(K) > f'(K)\left[f(K-1) + f(1)(\tilde{\delta}+\eta_i)\frac{\xi(\alpha_{CCP})}{\xi(\alpha_{uc})}H\right] \quad (173)$$

$$\Leftrightarrow \frac{f'(K-1)f(K) - f'(K)f(K-1)}{f'(K)f(1)}\frac{1}{(\tilde{\delta}+\eta_i)H} > \frac{\xi(\alpha_{CCP})}{\xi(\alpha_{uc})} \quad (174)$$

$$\Leftrightarrow \xi^{-1}\left(\frac{f'(K-1)f(K) - f'(K)f(K-1)}{f'(K)f(1)}\frac{1}{(\tilde{\delta}+\eta_i)H}\xi(\alpha_{uc})\right) < \alpha_{CCP}. \quad (175)$$

(c) The derivative with respect to β is equal to

$$\frac{\partial \Delta DL_i}{\partial \beta} = \frac{\partial}{\partial \beta}\frac{f(K-1)}{f(K)} + (\tilde{\delta}+\eta_i)\frac{\xi(\alpha_{CCP})}{\xi(\alpha_{uc})}H\frac{\partial}{\partial \beta}\frac{f(1)}{f(K)} < 0, \quad (176)$$

using Lemma IA.2.

□

Proposition 6 (Loss sharing based on net risk: directionality). *Assume that at least three entities have a portfolio that is not perfectly flat. Consider two entities $h, g \in \{1, ..., N\}, h \neq g$, with $G_h \geq G_g$. Then there exists $\varepsilon < 0$ such that the following holds: if entity h exhibits a lower portfolio directionality than g, $\eta_h < \eta_g$, and either $\eta_h = 0$ or $\eta_g < \eta_h + \varepsilon$, then the impact of central clearing on the expected default loss is smaller for h than for g,*

$$\Delta DL_h < \Delta DL_g. \quad (177)$$

Proof. Consider two different entities $h, g \in \{1, ..., N\}, h \neq g$. By assumption, there exists at least one other entity with positive net risk, $w \notin \{h, g\}$ with $G_w \eta_w > 0$. For $i \in \{h, g\}$, define

$$H_i = \mathbb{E}\left[\frac{\sum_{j=1,j\neq i}^{N} D_j G_j \eta_j}{(\tilde{\delta}+\eta_i)G_i + \sum_{j=1,j\neq i}^{N}(1-D_j)(\tilde{\delta}+\eta_j)G_j}\right] \tag{178}$$

$$= \mathbb{E}\left[\frac{1_{\{i=h\}}D_g G_g \eta_g + 1_{\{i=g\}}D_h G_h \eta_h + \sum_{j=1,j\notin\{h,g\}}^{N} D_j G_j \eta_j}{(1-1_{\{i=h\}}D_g)(\tilde{\delta}+\eta_g)G_g + (1-1_{\{i=g\}}D_h)(\tilde{\delta}+\eta_h)G_h + \sum_{j=1,j\notin\{h,g\}}^{N}(1-D_j)(\tilde{\delta}+\eta_j)G_j}\right] \tag{179}$$

$$= \mathbb{E}\left[\frac{\tilde{D}(1_{\{i=h\}}G_g\eta_g + 1_{\{i=g\}}G_h\eta_h) + A}{(1-1_{\{i=h\}}\tilde{D})(\tilde{\delta}+\eta_g)G_g + (1-1_{\{i=g\}}\tilde{D})(\tilde{\delta}+\eta_h)G_h + B}\right], \tag{180}$$

where we define by $\tilde{D} \sim Bern(\pi)$ a Bernoulli distributed random variable with success probability π that is independent from D_j for all $j \in \{1, ..., N\}\setminus\{h, g\}$, $A = \sum_{j=1,j\notin\{h,g\}}^{N} D_j G_j \eta_j$, and $B = \sum_{j=1,j\notin\{h,g\}}^{N}(1-D_j)(\tilde{\delta}+\eta_j)G_j$. Using Proposition 5, $\Delta DL_h < \Delta DL_g$ is equivalent to

$$\frac{f(K-1)}{f(K)} + (\tilde{\delta}+\eta_h)\frac{\xi(\alpha_{CCP})}{\xi(\alpha_{uc})}\frac{f(1)}{f(K)}\frac{1}{\pi}H_h - 1 < \frac{f(K-1)}{f(K)} + (\tilde{\delta}+\eta_g)\frac{\xi(\alpha_{CCP})}{\xi(\alpha_{uc})}\frac{f(1)}{f(K)}\frac{1}{\pi}H_g - 1 \tag{181}$$

$$\Leftrightarrow (\tilde{\delta}+\eta_h)H_h < (\tilde{\delta}+\eta_g)H_g \tag{182}$$

$$\Leftrightarrow (\tilde{\delta}+\eta_h)\mathbb{E}\left[\frac{\tilde{D}G_g\eta_g + A}{(\tilde{\delta}+\eta_h)G_h + (1-\tilde{D})(\tilde{\delta}+\eta_g)G_g + B}\right] \tag{183}$$

$$< (\tilde{\delta}+\eta_g)\mathbb{E}\left[\frac{\tilde{D}G_h\eta_h + A}{(\tilde{\delta}+\eta_g)G_g + (1-\tilde{D})(\tilde{\delta}+\eta_h)G_h + B}\right] \tag{184}$$

$$\Leftrightarrow \mathbb{E}\left[\frac{(\tilde{\delta}+\eta_h)(\tilde{D}G_g\eta_g + A)}{(\tilde{\delta}+\eta_h)G_h + (1-\tilde{D})(\tilde{\delta}+\eta_g)G_g + B} - \frac{(\tilde{\delta}+\eta_g)(\tilde{D}G_h\eta_h + A)}{(\tilde{\delta}+\eta_g)G_g + (1-\tilde{D})(\tilde{\delta}+\eta_h)G_h + B}\right] < 0 \tag{185}$$

$$\Leftrightarrow \mathbb{E}\Bigg[\frac{\substack{(\tilde{\delta}+\eta_h)(\tilde{D}G_g\eta_g+A)((\tilde{\delta}+\eta_g)G_g+(1-\tilde{D})(\tilde{\delta}+\eta_h)G_h+B) \\ -(\tilde{\delta}+\eta_g)(\tilde{D}G_h\eta_h+A)((\tilde{\delta}+\eta_h)G_h+(1-\tilde{D})(\tilde{\delta}+\eta_g)G_g+B)}}{\underbrace{((\tilde{\delta}+\eta_g)G_g + (1-\tilde{D})(\tilde{\delta}+\eta_h)G_h + B)((\tilde{\delta}+\eta_h)G_h + (1-\tilde{D})(\tilde{\delta}+\eta_g)G_g + B)}_{=C}}\Bigg] < 0. \tag{186}$$

The denominator is almost surely strictly positive since $\tilde{\delta} > 0$, $\eta_j \geq 0$, and $G_j > 0$ for all j. Assume that $\eta_h < \eta_g$ and $G_h \geq G_g$. Then, if $\tilde{\delta} = 0$, for the nominator it holds that

$$
\begin{aligned}
&\eta_h(\tilde{D}G_g\eta_g + A)(\eta_g G_g + (1-\tilde{D})\eta_h G_h + B) - \eta_g(\tilde{D}G_h\eta_h + A)(\eta_h G_h + (1-\tilde{D})\eta_g G_g + B)\\
=&A\left[\eta_h\left(\eta_g G_g + (1-\tilde{D})\eta_h G_h + B\right) - \eta_g\left(\eta_h G_h + (1-\tilde{D})\eta_g G_g + B\right)\right]\\
&+\tilde{D}\left[\eta_h G_g \eta_g\left(\eta_g G_g + (1-\tilde{D})\eta_h G_h + B\right) - \eta_g G_h \eta_h\left(\eta_h G_h + (1-\tilde{D})\eta_g G_g + B\right)\right]\\
=&A\left[\eta_h\left(\eta_g G_g + (1-\tilde{D})\eta_h G_h + B\right) - \eta_g\left(\eta_h G_h + (1-\tilde{D})\eta_g G_g + B\right)\right]\\
&+\eta_h\eta_g\tilde{D}\left[B\left(G_g - G_h\right) + (1-\tilde{D})G_h G_g\left(\eta_h - \eta_g\right) + \eta_g G_g^2 - \eta_h G_h^2\right]\\
\leq&A\left[B(\eta_h - \eta_g) + \eta_h\left(\eta_g G_g + (1-\tilde{D})\eta_h G_h\right) - \eta_g\left(\eta_h G_h + (1-\tilde{D})\eta_g G_g\right)\right]\\
&+\eta_h\eta_g\tilde{D}\left[B\left(G_g - G_h\right) + G_h^2(\eta_g - \eta_h)\right]\\
\leq&A\left[B(\eta_h - \eta_g) + \eta_h\eta_g\left(G_g - G_h\right) + (1-\tilde{D})\left((\eta_h)^2 G_h - (\eta_g)^2 G_g\right)\right] + \tilde{D}\eta_h\eta_g G_h^2(\eta_g - \eta_h)\\
\leq&A\left[(\eta_h)^2 G_h + \eta_h\eta_g\left(G_g - G_h\right) - (\eta_g)^2 G_g\right] + \tilde{D}G_h^2\eta_h\eta_g(\eta_g - \eta_h),
\end{aligned}
\tag{187}
$$

using that $\tilde{D} \in \{0, 1\}$ implies that $\tilde{D}(1-\tilde{D}) = 0$. Because for $x > 0$ it is

$$x^2 G_h + x\eta_g\left(G_g - G_h\right) - (\eta_g)^2 G_g < 0 \tag{188}$$

$$\Leftrightarrow x < \frac{-\eta_g\left(G_g - G_h\right) + \sqrt{(\eta_g)^2\left(G_g - G_h\right)^2 + 4G_h(\eta_g)^2 G_g}}{2G_h} \tag{189}$$

$$\Leftrightarrow x < \eta_g\frac{G_h - G_g + \sqrt{\left(G_h - G_g\right)^2 + 4G_h G_g}}{2G_h} \tag{190}$$

$$\Leftrightarrow x < \eta_g\frac{G_h - G_g + \sqrt{\left(G_h + G_g\right)^2}}{2G_h} \tag{191}$$

$$\Leftrightarrow x < \eta_g\frac{G_h - G_g + G_h + G_g}{2G_h} = \eta_g, \tag{192}$$

if $A > 0$, then it holds that

$$A\left[(\eta_h)^2 G_h + \eta_h\eta_g\left(G_g - G_h\right) - (\eta_g)^2 G_g\right] < 0. \tag{193}$$

Therefore, there exists $\varepsilon_1 > 0$ such that Expression (187) is strictly negative if $A > 0$ and $\eta_h\eta_g(\eta_g - \eta_h) < \varepsilon_1$. Because the nominator of C is continuous in $\tilde{\delta}$, there exists δ_0 such that the nominator of C is strictly negative if $A > 0$, $\eta_h\eta_g(\eta_g - \eta_h) < \varepsilon_1$, and $\tilde{\delta} < \delta_0$. Let $\tilde{\delta} \in (0, \delta_0)$. From the definition of A, $\pi > 0$, and the existence of an entity $w \notin \{h, g\}$ with $G_w\eta_w > 0$, it is $\mathbb{P}(A > 0) > \pi > 0$ and $\mathbb{P}(A < 0) = 0$. Therefore, there exists $0 < \varepsilon$ such that if either $\eta_h = 0$ or $\eta_g - \eta_h < \varepsilon$, then it holds that

$$\mathbb{E}[C] = \mathbb{P}(A=0)\mathbb{E}[C \mid A=0] + \mathbb{P}(A>0)\mathbb{E}[C \mid A>0] \tag{194}$$

$$\leq \mathbb{P}(A=0)\pi\mathbb{E}\left[\frac{G_h^2\eta_h\eta_g(\eta_g-\eta_h)}{((\tilde{\delta}+\eta_g)G_g+B)((\tilde{\delta}+\eta_h)G_h+B)}\right] + \underbrace{\underbrace{\mathbb{P}(A>0)}_{>0}\underbrace{\mathbb{E}[C \mid A>0]}_{<0}}_{<0} < 0, \tag{195}$$

and, thus, $\Delta DL_h < \Delta DL_g$. □

Proposition 7 (Loss sharing based on net risk in homogeneous networks)**.** *Consider a homogeneous network as in Assumption 1. Then, the impact of central clearing with loss sharing based on net risk on the expected default loss of entity i with* $\tilde{\delta} = 0$ *is equal to*

$$\Delta DL_i = \frac{f(K-1)}{f(K)} + \eta\frac{\xi(\alpha_{CCP})}{\xi(\alpha_{uc})}\frac{f(1)}{f(K)}\frac{1-\pi^{N-1}}{1-\pi} - 1, \tag{196}$$

where $f(K) = \sqrt{\beta^2\sigma_M^2K^2 + \sigma^2K}$*.* ΔDL_i *is*

(a) increasing with directionality, $\frac{\partial \Delta DL_i}{\partial \eta} > 0$*,*

(b) increasing with the number of derivative classes, $\frac{\partial \Delta DL_i}{\partial K} > 0$*, if, and only if,* $\eta < c$*, where* $c > 0$ *is a constant,*

(c) increasing with the probability of default, $\frac{\partial \Delta DL_i}{\partial \pi} > 0$*.*

Proof. Under Assumption 1, it is $G_i \equiv G > 0$ and $\eta_i \equiv \eta > 0$ for all $i = 1, ..., N,$. Then, the following identity holds:

$$\begin{aligned}
&\mathbb{E}\left[\frac{\sum_{j=1,j\neq i}^{N} D_j G_j \eta_j}{G_i(\tilde{\delta}+\eta_i)+\sum_{j=1,j\neq i}^{N}(1-D_j)G_j(\tilde{\delta}+\eta_j)}\right] && (197)\\
=&\mathbb{E}\left[\frac{G\eta\sum_{j=1,j\neq i}^{N} D_j}{G(\tilde{\delta}+\eta)+\sum_{j=1,j\neq i}^{N}(1-D_j)G(\tilde{\delta}+\eta)}\right] && (198)\\
=&\frac{\eta}{\tilde{\delta}+\eta}\mathbb{E}\left[\frac{\sum_{j=1,j\neq i}^{N} D_j-\sum_{j=1,j\neq i}^{N}(1-D_j)+\sum_{j=1,j\neq i}^{N}(1-D_j)}{1+\sum_{j=1,j\neq i}^{N}(1-D_j)}\right] && (199)\\
=&\frac{\eta}{\tilde{\delta}+\eta}\mathbb{E}\left[\frac{N-1-\sum_{j=1,j\neq i}^{N}(1-D_j)}{1+\sum_{j=1,j\neq i}^{N}(1-D_j)}\right] && (200)\\
=&\frac{\eta}{\tilde{\delta}+\eta}\mathbb{E}\left[\frac{N}{1+\sum_{j=1,j\neq i}^{N}(1-D_j)}-1\right] && (201)\\
=&\frac{N\eta}{\tilde{\delta}+\eta}\left(\mathbb{E}\left[\frac{1}{1+Y}\right]-\frac{1}{N}\right), && (202)
\end{aligned}$$

where $Y \sim Bin(N-1, 1-\pi)$. Using the properties of the Binomial distribution, it is

$$\mathbb{E}\left[\frac{1}{1+Y}\right]=\frac{1-\pi^N}{N(1-\pi)}. \tag{203}$$

Plugging into the formula in Proposition 5 yields

$$\begin{aligned}
\Delta DL_i &= \frac{f(K-1)}{f(K)}+(\tilde{\delta}+\eta_i)\frac{\xi(\alpha_{CCP})}{\xi(\alpha_{uc})}\frac{f(1)}{f(K)}\frac{1}{\pi}\frac{N\eta}{\tilde{\delta}+\eta}\left(\mathbb{E}\left[\frac{1}{1+Y}\right]-\frac{1}{N}\right)-1 && (204)\\
&= \frac{f(K-1)}{f(K)}+(\tilde{\delta}+\eta)\frac{\xi(\alpha_{CCP})}{\xi(\alpha_{uc})}\frac{f(1)}{f(K)}\frac{1}{\pi}\frac{N\eta}{\tilde{\delta}+\eta}\left(\frac{1-\pi^N}{N(1-\pi)}-\frac{1}{N}\right)-1 && (205)\\
&= \frac{f(K-1)}{f(K)}+(\tilde{\delta}+\eta)\frac{\xi(\alpha_{CCP})}{\xi(\alpha_{uc})}\frac{f(1)}{f(K)}\frac{1}{\pi}\frac{\eta}{\tilde{\delta}+\eta}\frac{1-\pi^N-1+\pi}{1-\pi}-1, && (206)\\
&= \frac{f(K-1)}{f(K)}+\eta\frac{\xi(\alpha_{CCP})}{\xi(\alpha_{uc})}\frac{f(1)}{f(K)}\frac{1-\pi^{N-1}}{1-\pi}-1. && (207)
\end{aligned}$$

(a) The derivative with respect to portfolio directionality η is equal to

$$\frac{\partial \Delta DL_i}{\partial \eta}=\frac{\xi(\alpha_{CCP})}{\xi(\alpha_{uc})}\frac{f(1)}{f(K)}\frac{1-\pi^{N-1}}{1-\pi}>0. \tag{208}$$

(b) The derivative with respect to the number of derivative classes K is:

$$\frac{\partial \Delta DL_i}{\partial K} = \frac{f'(K-1)f(K) - f'(K)f(K-1)}{f(K)^2} - \eta \frac{\xi(\alpha_{CCP})}{\xi(\alpha_{uc})} \frac{f(1)f'(K)}{f(K)^2} \frac{1-\pi^{N-1}}{1-\pi}, \tag{209}$$

which is positive if, and only if,

$$\eta \frac{\xi(\alpha_{CCP})}{\xi(\alpha_{uc})} \frac{f(1)f'(K)}{f(K)^2} \frac{1-\pi^{N-1}}{1-\pi} < \frac{f'(K-1)f(K) - f'(K)f(K-1)}{f(K)^2} \tag{210}$$

$$\Leftrightarrow \eta < \frac{f'(K-1)f(K) - f'(K)f(K-1)}{f(1)f'(K)} \frac{\xi(\alpha_{uc})}{\xi(\alpha_{CCP})} \frac{1-\pi}{1-\pi^{N-1}}, \tag{211}$$

where the right-hand side is strictly positive because $f'(\cdot) > 0$ and $f''(\cdot) < 0$ (see Lemma IA.2) imply that $f'(K-1)f(K) > f'(K)f(K-1)$.

(c) The derivative with respect to π is equal to

$$\frac{\partial \Delta DL_i}{\partial \pi} = \eta \frac{\xi(\alpha_{CCP})}{\xi(\alpha_{uc})} \frac{f(1)}{f(K)} \frac{-(N-1)\pi^{N-2}(1-\pi) - (-1)(1-\pi^{N-1})}{(1-\pi)^2} \tag{212}$$

$$= \eta \frac{\xi(\alpha_{CCP})}{\xi(\alpha_{uc})} \frac{f(1)}{f(K)} \frac{1-\pi^{N-1} - \pi^{N-2}(N-1) + \pi^{N-1}(N-1)}{(1-\pi)^2} \tag{213}$$

$$= \eta \frac{\xi(\alpha_{CCP})}{\xi(\alpha_{uc})} \frac{f(1)}{f(K)} \frac{1+\pi\pi^{N-2}(N-2) - \pi^{N-2}(N-1)}{(1-\pi)^2} \tag{214}$$

$$= \eta \frac{\xi(\alpha_{CCP})}{\xi(\alpha_{uc})} \frac{f(1)}{f(K)} \frac{1+\pi^{N-2}(\pi(N-1) - \pi - (N-1))}{(1-\pi)^2} \tag{215}$$

$$= \eta \frac{\xi(\alpha_{CCP})}{\xi(\alpha_{uc})} \frac{f(1)}{f(K)} \frac{1-\pi^{N-2}((N-1)(1-\pi)+\pi)}{(1-\pi)^2}. \tag{216}$$

Note that $g(N) = 1 - \pi^{N-2}((N-1)(1-\pi)+\pi)$ equals zero for $N = 2$, $g(2) = 1 - \pi^0(1 - \pi + \pi) = 1 - 1 = 0$, and that

$$g'(N) = -\log(\pi)\pi^{N-2}((N-1)(1-\pi)+\pi) - \pi^{N-2}(1-\pi) \tag{217}$$

$$= \pi^{N-2}(-\log(\pi)((N-1)(1-\pi)+\pi) - (1-\pi)), \tag{218}$$

which is strictly positive if, and only if,

$$-\log(\pi)((N-1)(1-\pi)+\pi) - (1-\pi) > 0 \tag{219}$$

$$\Leftrightarrow N-1 > \frac{1}{-\log(\pi)} - \frac{\pi}{1-\pi}. \tag{220}$$

It is $\frac{1}{-\log(\pi)} - \frac{\pi}{1-\pi} < 1 \Leftrightarrow \log(\pi) < \pi - 1$, which holds for all $\pi \in (0,1)$. Therefore,

$$\frac{1}{-\log(\pi)} - \frac{\pi}{1-\pi} < 1 \leq N - 1, \tag{221}$$

using that $N > 2$. Thus, $g'(N) > 0$, which, together with $g(2) = 0$, implies that $g(N) > 0$ for all $N \geq 2$. Therefore,

$$\frac{\partial \Delta DL_i}{\partial \pi} = \eta \frac{\xi(\alpha_{CCP})}{\xi(\alpha_{uc})} \frac{f(1)}{f(K)} \frac{g(N)}{(1-\pi)^2} > 0. \tag{222}$$

□

Proposition 8 (Loss sharing based on net risk in core-periphery networks)**.** *Consider a core-periphery network as in Assumption 2. Then, the impact of central clearing with loss sharing based on net risk as $\tilde{\delta}$ approaches 0 on the expected default loss of a peripheral entity $g \in \mathcal{N}_{per}$ is equal to*

$$\Delta DL_g = \frac{f(K-1)}{f(K)} + \frac{1-\pi^{2N/3-1}}{1-\pi} \frac{\xi(\alpha_{CCP})}{\xi(\alpha_{uc})} \frac{f(1)}{f(K)} - 1, \tag{223}$$

and for a core entity $h \in \mathcal{N}_{core}$ it is equal to

$$\Delta DL_h = \frac{f(K-1)}{f(K)} + \pi^{2N/3-1} \frac{6G_{per}}{(N-3)+6G_{per}} \frac{1-\pi^{N/3}}{1-\pi} \frac{\xi(\alpha_{CCP})}{\xi(\alpha_{uc})} \frac{f(1)}{f(K)} - 1, \tag{224}$$

where $f(K) = \sqrt{\beta^2\sigma_M^2 K^2 + \sigma^2 K}$.

For peripheral entities, central clearing is not beneficial, that is, $\Delta DL_g > 0$, if, and only if,

$$\frac{1-\pi^{2N/3-1}}{1-\pi} - \frac{\xi(\alpha_{uc})}{\xi(\alpha_{CCP})} \frac{f(K) - f(K-1)}{f(1)} > 0. \tag{225}$$

Holding all other parameters fixed,

(a) if $\alpha_{CCP} \leq \alpha_{uc}$, there exists $\hat{N} < \infty$ such that $\Delta DL_g > 0$ for all $N > \hat{N}$,

(b) there exists $\hat{K} < \infty$ such that $\Delta DL_g > 0$ for all $K > \hat{K}$,

(c) there exists $\hat{\alpha}_{uc} < 1$ such that $\Delta DL_g > 0$ for all $\alpha_{uc} > \hat{\alpha}_{uc}$.

For core entities $h \in \mathcal{N}_{core}$, central clearing is

- *beneficial, that is,* $\Delta DL_h < 0$*, if* $N > \hat{N}$ *for* $\hat{N} < \infty$*,*

- *and strictly more beneficial than for peripheral entities* $g \in \mathcal{N}_{per}$*,* $\Delta DL_h < \Delta DL_g$*.*

Proof. In the core-periphery network, the CCP's expected default loss per loss allocation unit is equal to

$$H_i = \mathbb{E}\left[\frac{\sum_{j=1,j\neq i}^{N} D_j G_j \eta_j}{(\tilde{\delta}+\eta_i)G_i + \sum_{j=1,j\neq i}^{N}(1-D_j)(\tilde{\delta}+\eta_j)G_j}\right] \tag{226}$$

$$= \mathbb{E}\left[\frac{\sum_{j\in\mathcal{N}_{per},j\neq i} D_j G_j \eta_j + \sum_{j\in\mathcal{N}_{core},j\neq i} D_j G_j \eta_j}{G_i(\tilde{\delta}+\eta_i) + \sum_{j\in\mathcal{N}_{per},j\neq i}(1-D_j)G_j(\tilde{\delta}+\eta_j) + \sum_{j\in\mathcal{N}_{core},j\neq i}(1-D_j)G_j(\tilde{\delta}+\eta_j)}\right] \tag{227}$$

$$= \mathbb{E}\left[\frac{G_{per}\sum_{j\in\mathcal{N}_{per},j\neq i} D_j}{G_i(\tilde{\delta}+\eta_i) + G_{per}\sum_{j\in\mathcal{N}_{per},j\neq i}(1-D_j)(\tilde{\delta}+1) + \tilde{\delta}G_{core}\sum_{j\in\mathcal{N}_{core},j\neq i}(1-D_j)}\right], \tag{228}$$

using that $\eta_j = 1$ if $j \in \mathcal{N}_{per}$ and $\eta_j = 0$ if $j \in \mathcal{N}_{core}$ by Assumption 2.
If $i \in \mathcal{N}_{per}$, then

$$H_i = \mathbb{E}\left[\frac{G_{per}\sum_{j\in\mathcal{N}_{per},j\neq i} D_j}{G_{per}(1+\tilde{\delta}) + G_{per}(1+\tilde{\delta})\sum_{j\in\mathcal{N}_{per},j\neq i}(1-D_j) + \tilde{\delta}G_{core}\sum_{j\in\mathcal{N}_{core}}(1-D_j)}\right]. \tag{229}$$

For $\tilde{\delta} = 0$ and $i \in \mathcal{N}_{per}$, H_i is equal to (note that the expectation is well-defined since $G_{per} > 0$)

$$H_i|_{\tilde{\delta}=0} = \mathbb{E}\left[\frac{G_{per}\sum_{j\in\mathcal{N}_{per},j\neq i} D_j}{G_{per} + G_{per}\sum_{j\in\mathcal{N}_{per},j\neq i}(1-D_j)}\right] \tag{230}$$

$$= \mathbb{E}\left[\frac{\sum_{j\in\mathcal{N}_{per},j\neq i} D_j}{1 + \sum_{j\in\mathcal{N}_{per},j\neq i}(1-D_j)}\right] \tag{231}$$

$$= \mathbb{E}\left[\frac{\sum_{j\in\mathcal{N}_{per},j\neq i} D_j + 1 + \sum_{j\in\mathcal{N}_{per},j\neq i}(1-D_j)}{1 + \sum_{j\in\mathcal{N}_{per},j\neq i}(1-D_j)} - 1\right] \tag{232}$$

$$= \mathbb{E}\left[\frac{\sum_{j\in\mathcal{N}_{per},j\neq i} 1 + 1}{1 + \sum_{j\in\mathcal{N}_{per},j\neq i}(1-D_j)} - 1\right] \tag{233}$$

$$= \mathbb{E}\left[\frac{|\mathcal{N}_{per}|}{1 + \sum_{j\in\mathcal{N}_{per},j\neq i}(1-D_j)} - 1\right] \tag{234}$$

$$= |\mathcal{N}_{per}|\mathbb{E}\left[\frac{1}{1 + \sum_{j\in\mathcal{N}_{per},j\neq i}(1-D_j)}\right] - 1 \tag{235}$$

$$= |\mathcal{N}_{per}|\frac{1-\pi^{|\mathcal{N}_{per}|}}{|\mathcal{N}_{per}|(1-\pi)} - 1 = \frac{1-\pi^{|\mathcal{N}_{per}|}}{1-\pi} - 1, \tag{236}$$

where in the last step we use the properties of the Binomial distribution. Using that $|\mathcal{N}_{per}| = \frac{2N}{3}$ is the number of entities in the periphery, applying the dominated convergence theorem, and plugging into the formula in Proposition 5 it is thus

$$\lim_{\tilde{\delta}\to 0} \Delta DL_i = \frac{f(K-1)}{f(K)} + \lim_{\tilde{\delta}\to 0}(\tilde{\delta} + \eta_i)\frac{\xi(\alpha_{CCP})}{\xi(\alpha_{uc})}\frac{f(1)}{f(K)}\frac{1}{\pi}H_i - 1 \tag{237}$$

$$= \frac{f(K-1)}{f(K)} + \frac{\xi(\alpha_{CCP})}{\xi(\alpha_{uc})}\frac{f(1)}{f(K)}\frac{1}{\pi}\left(\frac{1-\pi^{2N/3}}{1-\pi} - 1\right) - 1 \tag{238}$$

$$= \frac{f(K-1)}{f(K)} + \frac{\xi(\alpha_{CCP})}{\xi(\alpha_{uc})}\frac{f(1)}{f(K)}\frac{1}{\pi}\frac{1-\pi^{2N/3} - 1 + \pi}{1-\pi} - 1 \tag{239}$$

$$= \frac{f(K-1)}{f(K)} + \frac{\xi(\alpha_{CCP})}{\xi(\alpha_{uc})}\frac{f(1)}{f(K)}\frac{1-\pi^{2N/3-1}}{1-\pi} - 1. \tag{240}$$

Moreover,

$$\lim_{\tilde{\delta}\to 0} \Delta DL_i > 0 \tag{241}$$

$$\Leftrightarrow \frac{f(K-1)}{f(K)} + \frac{\xi(\alpha_{CCP})}{\xi(\alpha_{uc})} \frac{f(1)}{f(K)} \frac{1-\pi^{2N/3-1}}{1-\pi} - 1 > 0 \tag{242}$$

$$\Leftrightarrow \frac{\xi(\alpha_{CCP})}{\xi(\alpha_{uc})} \frac{f(1)}{f(K)} \frac{1-\pi^{2N/3-1}}{1-\pi} - \frac{f(K)-f(K-1)}{f(K)} > 0 \tag{243}$$

$$\Leftrightarrow \underbrace{\frac{1-\pi^{2N/3-1}}{1-\pi} - \frac{\xi(\alpha_{uc})}{\xi(\alpha_{CCP})} \frac{f(K)-f(K-1)}{f(1)}}_{A} > 0. \tag{244}$$

(a) A is increasing with N since $\frac{\partial A}{\partial N} = (-\log(\pi)) \frac{\frac{2}{3}\pi^{2N/3-1}}{1-\pi} > 0$, and it is

$$\lim_{N\to\infty} A = \frac{1}{1-\pi} - \frac{\xi(\alpha_{uc})}{\xi(\alpha_{CCP})} \frac{f(K)-f(K-1)}{f(1)}, \tag{245}$$

which is positive if, and only if,

$$\frac{1}{1-\pi} > \frac{\xi(\alpha_{uc})}{\xi(\alpha_{CCP})} \frac{f(K)-f(K-1)}{f(1)} \tag{246}$$

$$\Leftrightarrow \pi > 1 - \underbrace{\frac{f(1)}{f(K)-f(K-1)}}_{>1} \frac{\xi(\alpha_{CCP})}{\xi(\alpha_{uc})}. \tag{247}$$

Note that $\frac{f(1)}{f(K)-f(K-1)} = 1$ for $K = 1$ and $\frac{f(1)}{f(K)-f(K-1)} > 1$ for all $K > 1$ since $f(K) - f(K-1)$ is decreasing with K (see Lemma IA.2). Since $\xi(\alpha)$ is decreasing with α (see Lemma IA.1), if $\alpha_{CCP} \leq \alpha_{uc}$, it is $\frac{\xi(\alpha_{CCP})}{\xi(\alpha_{uc})} \geq 1$ and $1 - \frac{f(1)}{f(K)-f(K-1)} \frac{\xi(\alpha_{CCP})}{\xi(\alpha_{uc})} < 0$. In this case, $\lim_{N\to\infty} A > 0$. Therefore, if $\alpha_{CCP} \leq \alpha_{uc}$, there exists $\hat{N} < \infty$ such that $\lim_{\tilde{\delta}\to 0} \Delta DL_i > 0$ for all $N > \hat{N}$.

(b) A is increasing with K and it is

$$\lim_{K\to\infty} A = \frac{1-\pi^{2N/3-1}}{1-\pi} > 0, \tag{248}$$

since $\frac{2N}{3} > 1$. Thus, there exists $\hat{K} < \infty$ such that $\lim_{\tilde{\delta}\to 0} \Delta DL_i > 0$ for all $K > \hat{K}$.

(c) Since $\xi(\alpha)$ is decreasing with α and $\lim_{\alpha\to 1}\xi(\alpha) = 0$ and $\xi(0.5) = \varphi(0)$, it is

$$\lim_{\alpha_{uc}\to 1} A = \frac{1-\pi^{2N/3-1}}{1-\pi} > 0, \tag{249}$$

and, thus, there exists $\hat{\alpha}_{uc} < 1$ such that $\lim_{\tilde{\delta}\to 0}\Delta DL_i > 0$ for all $\alpha_{uc} > \hat{\alpha}_{uc}$.

If $i \in \mathcal{N}_{core}$, then

$$\tilde{\delta}H_i = \mathbb{E}\left[\frac{\tilde{\delta}G_{per}\sum_{j\in\mathcal{N}_{per}} D_j}{G_{core}\tilde{\delta} + G_{per}\sum_{j\in\mathcal{N}_{per}}(1-D_j)(1+\tilde{\delta}) + G_{core}\sum_{j\in\mathcal{N}_{core},j\neq i}(1-D_j)\tilde{\delta}}\right] \tag{250}$$

$$=\mathbb{E}\left[\frac{\tilde{\delta}G_{per}\sum_{j\in\mathcal{N}_{per}} D_j}{G_{core}\tilde{\delta} + G_{per}\sum_{j\in\mathcal{N}_{per}}(1-D_j)(1+\tilde{\delta}) + \tilde{\delta}G_{core}\sum_{j\in\mathcal{N}_{core},j\neq i}(1-D_j)}\right] \tag{251}$$

$$=\mathbb{P}(\mathcal{D}_{per})\mathbb{E}\left[\frac{\tilde{\delta}G_{per}\sum_{j\in\mathcal{N}_{per}} D_j}{G_{core}\tilde{\delta} + G_{per}\sum_{j\in\mathcal{N}_{per}}(1-D_j)(1+\tilde{\delta}) + \tilde{\delta}G_{core}\sum_{j\in\mathcal{N}_{core},j\neq i}(1-D_j)} \mid \mathcal{D}_{per}\right] \tag{252}$$

$$+ (1-\mathbb{P}(\mathcal{D}_{per}))\mathbb{E}\left[\frac{\tilde{\delta}G_{per}\sum_{j\in\mathcal{N}_{per}} D_j}{G_{core}\tilde{\delta} + G_{per}\sum_{j\in\mathcal{N}_{per}}(1-D_j)(1+\tilde{\delta}) + \tilde{\delta}G_{core}\sum_{j\in\mathcal{N}_{core},j\neq i}(1-D_j)} \mid \overline{\mathcal{D}_{per}}\right]$$

$$=\mathbb{P}(\mathcal{D}_{per})\underbrace{\mathbb{E}\left[\frac{\tilde{\delta}G_{per}\sum_{j\in\mathcal{N}_{per}} 1}{G_{core}\tilde{\delta} + \tilde{\delta}G_{core}\sum_{j\in\mathcal{N}_{core},j\neq i}(1-D_j)}\right]}_{=A_1} \tag{253}$$

$$+ (1-\mathbb{P}(\mathcal{D}_{per}))\underbrace{\mathbb{E}\left[\frac{\tilde{\delta}G_{per}\sum_{j\in\mathcal{N}_{per}} D_j}{G_{core}\tilde{\delta} + G_{per}\sum_{j\in\mathcal{N}_{per}}(1-D_j)(1+\tilde{\delta}) + \tilde{\delta}G_{core}\sum_{j\in\mathcal{N}_{core},j\neq i}(1-D_j)} \mid \overline{\mathcal{D}_{per}}\right]}_{=A_2},$$

using that D_n and D_m are independently distributed for $n \neq m$, where $\mathcal{D}_{per} = \{D \in \{0,1\}^N : D_j = 1\ \forall j \in \mathcal{N}_{per}\}$ is the set of states in which all peripheral entities default and $\overline{\mathcal{D}_{per}}$ its complement. Since conditional on $\overline{\mathcal{D}_{per}}$, there exists $j \in \mathcal{N}_{per}$ such that $(1-D_j)(1+\tilde{\delta}) = 1+\tilde{\delta} > 0$, A_2 almost surely has a strictly positive denominator and is, thus, well-defined for $\tilde{\delta} = 0$, which implies that (using the dominated convergence theorem)

$$\lim_{\tilde{\delta}\to 0} A_2 = 0.$$

Moreover, for all $\tilde{\delta} > 0$, it is

$$A_1 = \frac{|\mathcal{N}_{per}|G_{per}}{G_{core}} \mathbb{E}\left[\frac{1}{1+\sum_{j\in\mathcal{N}_{core}, j\neq i}(1-D_j)}\right] = \frac{|\mathcal{N}_{per}|G_{per}}{G_{core}} \frac{1-\pi^{|\mathcal{N}_{core}|}}{|\mathcal{N}_{core}|(1-\pi)} \tag{254}$$

$$= \frac{\frac{2N}{3}G_{per}}{\frac{N-3}{3}+2G_{per}} \frac{1-\pi^{N/3}}{N/3(1-\pi)} = \frac{6G_{per}}{(N-3)+6G_{per}} \frac{1-\pi^{N/3}}{1-\pi}, \tag{255}$$

using that $N_{core} = \frac{N-3}{3} + 2G_{per}$, $|\mathcal{N}_{per}| = \frac{2N}{3}$, and $|\mathcal{N}_{core}| = \frac{N}{3}$ and the properties of the Binomial distribution. Therefore,

$$\lim_{\tilde{\delta}\to 0} \tilde{\delta} H_i = \mathbb{P}(\mathcal{D}_{per}) \lim_{\tilde{\delta}\to 0} A_1 + (1-\mathbb{P}(\mathcal{D}_{per})) \lim_{\tilde{\delta}\to 0} A_2 \tag{256}$$

$$= \pi^{2N/3} \frac{6G_{per}}{(N-3)+6G_{per}} \frac{1-\pi^{N/3}}{1-\pi} \tag{257}$$

and

$$\lim_{\tilde{\delta}\to 0} \Delta DL_i = \frac{f(K-1)}{f(K)} + \lim_{\tilde{\delta}\to 0} \tilde{\delta} H_i \frac{\xi(\alpha_{CCP})}{\xi(\alpha_{uc})} \frac{f(1)}{f(K)} \frac{1}{\pi} - 1 \tag{258}$$

$$= \frac{f(K-1)}{f(K)} + \pi^{2N/3} \frac{6G_{per}}{(N-3)+6G_{per}} \frac{1-\pi^{N/3}}{1-\pi} \frac{\xi(\alpha_{CCP})}{\xi(\alpha_{uc})} \frac{f(1)}{f(K)} \frac{1}{\pi} - 1 \tag{259}$$

$$= \frac{f(K-1)}{f(K)} + \pi^{2N/3} \frac{6G_{per}}{(N-3)+6G_{per}} \frac{1-\pi^{N/3}}{\pi(1-\pi)} \frac{\xi(\alpha_{CCP})}{\xi(\alpha_{uc})} \frac{f(1)}{f(K)} - 1. \tag{260}$$

Consequently, $\lim_{N\to\infty} \lim_{\tilde{\delta}\to 0} \Delta DL_i = \frac{f(K-1)}{f(K)} - 1 < 0$. Therefore, there exists $\hat{N}$ such that $\lim_{\tilde{\delta}\to 0} \Delta DL_i < 0$ for all $N > \hat{N}$, that is, such that entities in the core benefit from central clearing.

For $g \in \mathcal{N}_{per}$ and $h \in \mathcal{N}_{core}$ it is

$$\lim_{\tilde{\delta}\to 0} \Delta DL_g > \lim_{\tilde{\delta}\to 0} \Delta DL_h \tag{261}$$

$$\Leftrightarrow \frac{1-\pi^{2N/3-1}}{1-\pi} \frac{\xi(\alpha_{CCP})}{\xi(\alpha_{uc})} \frac{f(1)}{f(K)} > \pi^{2N/3-1} \frac{6G_{per}}{(N-3)+6G_{per}} \frac{1-\pi^{N/3}}{1-\pi} \frac{\xi(\alpha_{CCP})}{\xi(\alpha_{uc})} \frac{f(1)}{f(K)} \tag{262}$$

$$\Leftrightarrow \frac{1-\pi^{2N/3-1}}{1-\pi} > \pi^{2N/3-1} \frac{6G_{per}}{(N-3)+6G_{per}} \frac{1-\pi^{N/3}}{1-\pi}, \tag{263}$$

which holds because

$$\pi^{2N/3-1} \underbrace{\frac{6G_{per}}{(N-3)+6G_{per}}}_{\leq 1} \frac{1-\pi^{N/3}}{1-\pi} \leq \pi^{N/3-1} \frac{1-\pi^{N/3}}{1-\pi} \tag{264}$$

$$= \frac{\pi^{N/3-1} - \pi^{N/3-1}\pi^{N/3}}{1-\pi} < \frac{1-\pi^{2N/3-1}}{1-\pi}. \tag{265}$$

□

Example 1. *Consider a core-periphery network. Central clearing with loss sharing based on net risk reduces the expected default loss in aggregate, but it does not reduce the expected default loss of peripheral entities for the following parameters:* $G_{per} = 1$, $\pi = 0.05$, $N = 21$, $K = 10$, $\alpha_{uc} = \alpha_{CCP} = 0.99$, $\sigma = \sigma_M = 1$, $\beta = 0.3$.

Figure 3 illustrates comparative statics varying either the number of market participants, N*, or the systematic risk exposure,* β*, while holding all other parameters constant. Figure 3, panel A, shows that larger* N *reduces* ΔADL*. Intuitively, a larger market enables more risk sharing and, thus, central clearing reduces the expected aggregate default loss by more. In other words, central clearing becomes more beneficial overall. However, the impact of central clearing on an individual entity's expected default loss is largely unaffected by* N*. This is intuitive from the closed-form expressions in Proposition 8. A larger expected number of defaulters roughly balances a larger expected number of survivors.*

Figure 3, panel B, shows that a larger systematic risk exposure β *reduces* ΔADL *as well as each entity's* ΔDL*. This result is in line with Proposition 5, which shows that larger* β *reduces bilateral netting efficiency and, thereby, makes central clearing relatively more beneficial. This effect is particularly pronounced for peripheral entities because they make larger loss sharing contributions.*

Proof. From Proposition 3, the impact of clearing on the expected aggregate default loss is equal to

$$\Delta ADL = \frac{\xi(\alpha_{CCP})}{\xi(\alpha_{uc})} \frac{f(1)}{f(K)} \eta_{agg} + \frac{f(K-1)}{f(K)} - 1, \tag{266}$$

where

$$\eta_{agg} = \frac{\sum_{i=1}^{N} \left| \sum_{j \in \mathcal{N}_i} v_{ij} \right|}{\sum_{i=1}^{N} G_i} = \frac{\frac{2N}{3} G_{per} + \frac{N}{3} \cdot 0}{\frac{2N}{3} G_{per} + \frac{N}{3} \frac{N-3+6G_{per}}{3}} \tag{267}$$

$$= \frac{6G_{per}}{6G_{per} + N - 3 + 6G_{per}} = \frac{6G_{per}}{12G_{per} + N - 3} \tag{268}$$

in the case of a core-periphery network. The statement follows from setting the variables equal to the parameters. □

Proposition 9 (Loss sharing based on net and gross risk)**.** *Consider loss sharing rules based on net and gross risk, that is, with $\delta \in (0,1)$.*

(a) Assume that $\eta_j = \eta \in [0,1]$ for all $j = 1, ..., N$. Then, for any $i \in \{1, ..., N\}$, it is $\frac{\partial \Delta DL_i}{\partial \delta} = 0$.

(b) Consider an entity with a flat portfolio, $\eta_i = 0$. Assume that there exist at least two fellow clearing members a and b, $a \neq b$, with portfolio directionality $\eta_a > 0$ and $\eta_b > 0$. Then,

$$\frac{\partial \Delta DL_i}{\partial \delta} > 0.$$

(c) Consider an entity with a fully directional portfolio, $\eta_i = 1$. Assume that there exist at least two fellow clearing members a and b, $a \neq b$, with portfolio directionality $\eta_a < 1$ and $\eta_b > 0$. Then,

$$\frac{\partial \Delta DL_i}{\partial \delta} < 0.$$

Proof. From Definition 4 and Proposition 1, it is

$$w_i(\delta) = \delta G_i f(1) + (1-\delta)\eta_i G_i f(1) = (\delta + (1-\delta)\eta_i) G_i f(1). \tag{269}$$

The derivative of $w_i(\delta)$ with respect to δ is equal to

$$\frac{\partial w_i}{\partial \delta} = (1-\eta_i) G_i f(1). \tag{270}$$

Define by $H = \frac{\sum_{j=1, j\neq i}^{N} D_j G_j \eta_j}{w_i(\delta) + \sum_{j=1, j\neq i}^{N} (1-D_j) w_j(\delta)}$ the CCP's default losses per unit of loss sharing weight. The derivative of ΔDL_i with respect to δ is equal to

$$\frac{\partial \Delta DL_i}{\partial \delta} = \frac{f(1)}{G_i \pi f(K)} \frac{\xi(\alpha_{CCP})}{\xi(\alpha_{uc})} \frac{\partial}{\partial \delta} w_i(\delta) \mathbb{E}\left[\frac{\sum_{j=1, j\neq i}^N D_j G_j \eta_j}{w_i(\delta) + \sum_{j=1,j\neq i}^N (1-D_j) w_j(\delta)}\right] \tag{271}$$

$$= \frac{f(1)}{G_i \pi f(K)} \frac{\xi(\alpha_{CCP})}{\xi(\alpha_{uc})} \Bigg((1-\eta_i) G_i f(1) \mathbb{E}[H] \tag{272}$$

$$- w_i(\delta) \mathbb{E}\left[H \frac{(1-\eta_i) G_i f(1) + \sum_{j=1,j\neq i}^N (1-D_j)(1-\eta_j) G_j f(1)}{G_i f(1)(\delta + (1-\delta)\eta_i) + \sum_{j=1,j\neq i}^N (1-D_j) w_j(\delta)} \right] \Bigg) \tag{273}$$

$$= \frac{f(1)}{\pi f(K)} \frac{\xi(\alpha_{CCP})}{\xi(\alpha_{uc})} \Bigg((1-\eta_i) f(1) \mathbb{E}[H] \tag{274}$$

$$- f(1)(\delta + (1-\delta)\eta_i) \mathbb{E}\left[H \frac{(1-\eta_i) G_i + \sum_{j=1,j\neq i}^N (1-D_j)(1-\eta_j) G_j}{G_i(\delta + (1-\delta)\eta_i) + \sum_{j=1,j\neq i}^N (1-D_j)(\delta + (1-\delta)\eta_j) G_j} \right] \Bigg), \tag{275}$$

which is positive if, and only if,

$$\frac{1-\eta_i}{\delta + (1-\delta)\eta_i} \mathbb{E}[H] > \mathbb{E}\left[H \frac{(1-\eta_i) G_i + \sum_{j=1,j\neq i}^N (1-D_j)(1-\eta_j) G_j}{G_i(\delta + (1-\delta)\eta_i) + \sum_{j=1,j\neq i}^N (1-D_j)(\delta + (1-\delta)\eta_j) G_j} \right] \tag{276}$$

$$\Leftrightarrow \frac{1-\eta_i}{\delta + (1-\delta)\eta_i} \mathbb{E}[H] > \mathbb{E}\left[H \frac{1}{\delta} \left(1 - \frac{w_i(0) + \sum_{j=1,j\neq i}^N (1-D_j) w_j(0)}{w_i(\delta) + \sum_{j=1,j\neq i}^N (1-D_j) w_j(\delta)} \right) \right] \tag{277}$$

$$\Leftrightarrow \delta \frac{1-\eta_i}{\delta + (1-\delta)\eta_i} \mathbb{E}[H] > \mathbb{E}[H] - \mathbb{E}\left[H \frac{w_i(0) + \sum_{j=1,j\neq i}^N (1-D_j) w_j(0)}{w_i(\delta) + \sum_{j=1,j\neq i}^N (1-D_j) w_j(\delta)} \right] \tag{278}$$

$$\Leftrightarrow \mathbb{E}\left[H \frac{w_i(0) + \sum_{j=1,j\neq i}^N (1-D_j) w_j(0)}{w_i(\delta) + \sum_{j=1,j\neq i}^N (1-D_j) w_j(\delta)} \right] > \mathbb{E}[H] - \delta \frac{1-\eta_i}{\delta + (1-\delta)\eta_i} \mathbb{E}[H] \tag{279}$$

$$\Leftrightarrow \mathbb{E}\left[H \frac{w_i(0) + \sum_{j=1,j\neq i}^N (1-D_j) w_j(0)}{w_i(\delta) + \sum_{j=1,j\neq i}^N (1-D_j) w_j(\delta)} \right] > \mathbb{E}\left[H \frac{w_i(0)}{w_i(\delta)} \right] \tag{280}$$

$$\Leftrightarrow \mathbb{E}\left[H \left(\frac{w_i(0) + \sum_{j=1,j\neq i}^N (1-D_j) w_j(0)}{w_i(\delta) + \sum_{j=1,j\neq i}^N (1-D_j) w_j(\delta)} - \frac{w_i(0)}{w_i(\delta)} \right) \right] > 0 \tag{281}$$

$$\Leftrightarrow \mathbb{E}\left[H \frac{\sum_{j=1,j\neq i}^N (1-D_j)(w_j(0) w_i(\delta) - w_j(\delta) w_i(0))}{w_i(\delta)\left(w_i(\delta) + \sum_{j=1,j\neq i}^N (1-D_j) w_j(\delta) \right)} \right] > 0 \tag{282}$$

$$\Leftrightarrow \mathbb{E}\left[\tilde{H} \sum_{j=1,j\neq i}^N (1-D_j)(w_j(0) w_i(\delta) - w_j(\delta) w_i(0)) \right] > 0, \tag{283}$$

where we define $\tilde{H} = \frac{h}{w_i(\delta)\left(w_i(\delta) + \sum_{j=1,j\neq i}^N (1-D_j) w_j(\delta)\right)}$, which is nonnegative with probability one. From Inequality (283) it follows that:

(a) $\frac{\partial \Delta DL_i}{\partial \delta} = 0$ if $\eta_j \equiv \eta \in [0,1]$ for all $j = 1, ..., N$, since in this case

$$\mathbb{E}\left[\tilde{H} \sum_{j=1, j\neq i}^{N} (1 - D_j)(w_j(0)w_i(\delta) - w_j(\delta)w_i(0))\right] \tag{284}$$

$$= f(1)\mathbb{E}\left[\tilde{H} \sum_{j=1, j\neq i}^{N} (1 - D_j)(\eta G_j w_i(\delta) - w_j(\delta)\eta G_i)\right] \tag{285}$$

$$= f(1)\eta\mathbb{E}\left[\tilde{H} \sum_{j=1, j\neq i}^{N} (1 - D_j)(G_j(\delta + (1-\delta)\eta)G_i f(1) - (\delta + (1-\delta)\eta)G_j f(1) G_i)\right] \tag{286}$$

$$= f(1)^2\eta(\delta + (1-\delta)\eta)G_i\mathbb{E}\left[\tilde{H} \sum_{j=1, j\neq i}^{N} (1 - D_j)(G_j - G_j)\right] = 0. \tag{287}$$

(b) $\frac{\partial \Delta DL_i}{\partial \delta} > 0$ if $\eta_i = 0$ since in this case $w_i(0) = \eta_i f(1) G_i = 0$ and, thus,

$$\mathbb{E}\left[\tilde{H} \sum_{j=1, j\neq i}^{N} (1 - D_j)(w_j(0)w_i(\delta) - w_j(\delta)w_i(0))\right] \tag{288}$$

$$= \mathbb{E}\left[\tilde{H} \sum_{j=1, j\neq i}^{N} (1 - D_j)(\eta_j f(1) G_j w_i(\delta))\right] \tag{289}$$

$$\geq w_i(\delta) f(1)\mathbb{E}\left[\tilde{H}\left((1 - D_a)\eta_a G_a + (1 - D_b)\eta_b G_b\right)\right] > 0, \tag{290}$$

where we use that by assumption there exist $a, b \in \{1, ..., N\} \backslash \{i\}, a \neq b$, with $\eta_a > 0$ and $\eta_b > 0$ such that $\mathbb{P}(D_a = 1, D_b = 0) + \mathbb{P}(D_a = 0, D_b = 1) > 0$ implies that $\mathbb{P}(\tilde{h} > 0, (1 - D_a)\eta_a G_a + (1 - D_b)\eta_b G_b > 0) > 0$.

(c) $\frac{\partial \Delta DL_i}{\partial \delta} < 0$ if $\eta_i = 1$ since in this case $w_i(\delta) = (\delta + (1-\delta)) f(1) G_i \equiv f(1) G_i$ and, thus,

$$\mathbb{E}\left[\tilde{H} \sum_{j=1, j\neq i}^{N} (1 - D_j)(w_j(0)w_i(\delta) - w_j(\delta)w_i(0))\right] \tag{291}$$

$$= f(1)G_i\mathbb{E}\left[\tilde{H} \sum_{j=1, j\neq i}^{N} (1 - D_j)(w_j(0) - w_j(\delta))\right] \tag{292}$$

$$\leq f(1)G_i\mathbb{E}\left[\tilde{H}(1 - D_a)(w_a(0) - w_a(\delta))\right] < 0, \tag{293}$$

where we use that by assumption there exist $a, b \in \{1, ..., N\} \backslash \{i\}, a \neq b$, with $\eta_a < 1$ and $\eta_b > 0$ such that $w_a(0) - w_a(\delta) = (\eta_a - (\delta + (1-\delta)\eta_a)) f(1) G_a = -\delta(1 - \eta_a) f(1) G_a < 0$ for all $\delta > 0$ and that $P(D_b \eta_b G_b > 0, D_a = 0) > 0$, implying that $\mathbb{P}(\tilde{h} > 0, (1 - D_a)(w_a(0) - w_a(\delta)) <$

$0) > 0$ and $\mathbb{P}(\tilde{h} < 0, (1 - D_a)(w_a(0) - w_a(\delta)) < 0) = 0$.

□

Proposition 10 (Loss sharing based on gross risk)**.** *Consider two entities g, h, $g \neq h$, and assume that loss sharing is proportional to gross portfolio risk, $\delta = 1$. Then, the difference in the impact of central clearing between the two entities is equal to*

$$\Delta DL_g - \Delta DL_h \tag{294}$$
$$= \frac{\xi(\alpha_{CCP})}{\xi(\alpha_{uc})} \frac{f(1)}{f(K)} \frac{1}{\pi} \left(\mathbb{E}\left[\frac{\sum_{j=1}^N D_j G_j \eta_j}{\sum_{j=1}^N (1 - D_j) G_j} \mid D_g = 0 \right] - \mathbb{E}\left[\frac{\sum_{j=1}^N D_j G_j \eta_j}{\sum_{j=1}^N (1 - D_j) G_j} \mid D_h = 0 \right] \right).$$

(a) Conditional on $D_g = D_h$, the impact of central clearing is the same across entities:

$$\Delta DL_{g|D_g = D_h} = \Delta DL_{h|D_g = D_h}. \tag{295}$$

(b) If $\eta_g = \eta_h$, then

$$G_h > G_g \;\Rightarrow\; \Delta DL_h < \Delta DL_g. \tag{296}$$

(c) If $G_g = G_h$, then

$$\eta_h > \eta_g \;\Leftrightarrow\; \Delta DL_h < \Delta DL_g. \tag{297}$$

(d) If $h \in \mathcal{N}_{core}$ and $g \in \mathcal{N}_{per}$ in a core-periphery network, then there exists $\hat{\pi} > 0$ such that for all $\pi \in (0, \hat{\pi})$ it is

$$\Delta DL_g < \Delta DL_h. \tag{298}$$

Proof. When $\delta = 1$, loss sharing weights are equal to $w_i = G_i f(1)$. Using Proposition 4, the impact of central clearing on i's expected default loss is then given by

$$\Delta DL_i = \frac{f(K-1)}{f(K)} + \frac{w_i(\delta)}{G_i} \frac{\xi(\alpha_{CCP})}{\xi(\alpha_{uc})} \frac{f(1)}{f(K)} \frac{1}{\pi} \mathbb{E}\left[\frac{\sum_{j=1, j\neq i}^N D_j G_j \eta_j}{w_i(\delta) + \sum_{j=1, j\neq i}^N (1 - D_j) w_j(\delta)} \right] - 1 \tag{299}$$

$$= \frac{f(K-1)}{f(K)} + \frac{G_i f(1)}{G_i} \frac{\xi(\alpha_{CCP})}{\xi(\alpha_{uc})} \frac{f(1)}{f(K)} \frac{1}{\pi} \mathbb{E}\left[\frac{\sum_{j=1, j\neq i}^N D_j G_j \eta_j}{G_i f(1) + \sum_{j=1, j\neq i}^N (1 - D_j) G_j f(1)} \right] - 1 \tag{300}$$

$$= \frac{f(K-1)}{f(K)} + \frac{\xi(\alpha_{CCP})}{\xi(\alpha_{uc})} \frac{f(1)}{f(K)} \frac{1}{\pi} \mathbb{E}\left[\frac{\sum_{j=1}^N D_j G_j \eta_j}{\sum_{j=1}^N (1 - D_j) G_j} \mid D_i = 0 \right] - 1. \tag{301}$$

Consider two entities $g, h \in \{1, ..., N\}, g \neq h$. Then, the difference in the impact of central clearing is equal to

$$\begin{aligned} & \Delta DL_g - \Delta DL_h \\ =& \frac{\xi(\alpha_{CCP})}{\xi(\alpha_{uc})} \frac{f(1)}{f(K)} \frac{1}{\pi} \left(\mathbb{E}\left[\frac{\sum_{j=1}^N D_j G_j \eta_j}{\sum_{j=1}^N (1-D_j) G_j} \mid D_g = 0 \right] - \mathbb{E}\left[\frac{\sum_{j=1}^N D_j G_j \eta_j}{\sum_{j=1}^N (1-D_j) G_j} \mid D_h = 0 \right] \right). \end{aligned} \tag{302}$$

Define by $\tilde{D}$ a Bernoulli distributed random variable with success probability π such that $\tilde{D}$ and D_j are independently distributed for all $j \notin \{g, h\}$. With $A = \sum_{j=1, j \notin \{g,h\}}^N D_j G_j \eta_j \geq 0$ and $B = \sum_{j=1, j \notin \{g,h\}}^N (1-D_j) G_j \geq 0$

$$\frac{\Delta DL_g - \Delta DL_h}{\frac{\xi(\alpha_{CCP})}{\xi(\alpha_{uc})} \frac{f(1)}{f(K)} \frac{1}{\pi}} \tag{303}$$

$$= \mathbb{E}\left[\frac{\tilde{D} G_h \eta_h + A}{G_g + (1-\tilde{D}) G_h + B} - \frac{\tilde{D} G_g \eta_g + A}{G_h + (1-\tilde{D}) G_g + B} \right] \tag{304}$$

$$= \mathbb{E}\left[\frac{(\tilde{D} G_h \eta_h + A)(G_h + (1-\tilde{D}) G_g + B) - (\tilde{D} G_g \eta_g + A)(G_g + (1-\tilde{D}) G_h + B)}{(G_g + (1-\tilde{D}) G_h + B)(G_h + (1-\tilde{D}) G_g + B)} \right] \tag{305}$$

$$= \mathbb{E}\left[\frac{A \tilde{D} [G_h - G_g] + \tilde{D} \left[G_h^2 \eta_h - G_g^2 \eta_g + G_g G_h (1-\tilde{D})(\eta_h - \eta_g) + B(G_h \eta_h - G_g \eta_g) \right]}{(G_g + (1-\tilde{D}) G_h + B)(G_h + (1-\tilde{D}) G_g + B)} \right] \tag{306}$$

$$= \mathbb{E}\left[\tilde{D} \frac{A(G_h - G_g) + G_h^2 \eta_h - G_g^2 \eta_g + B(G_h \eta_h - G_g \eta_g)}{(G_g + (1-\tilde{D}) G_h + B)(G_h + (1-\tilde{D}) G_g + B)} \right] \tag{307}$$

$$= \pi \mathbb{E}\left[\frac{A(G_h - G_g) + G_h^2 \eta_h - G_g^2 \eta_g + B(G_h \eta_h - G_g \eta_g)}{(G_g + B)(G_h + B)} \right], \tag{308}$$

using that

$$\tilde{D}(1-\tilde{D}) = \begin{cases} 0 \times (1-0) = 0, \text{ if } \tilde{D} = 0 \\ 1 \times (1-1) = 0, \text{ if } \tilde{D} = 1. \end{cases} \tag{309}$$

(a) If $D_g = 0$ and $D_h = 0$, then Equation (302) implies that the impact of central clearing is the same for entities h and g. Moreover, if $D_g = 1$ and $D_h = 1$, cleared and uncleared default losses are zero and the impact of central clearing coincides, as well. Therefore, conditional on $D_g = D_h$, the impact of central clearing is the same across entities, $\Delta DL_{g|D_g = D_h} = \Delta DL_{h|D_g = D_h}$.

(b) If $\eta_g = \eta_h$, then Expression (308) is equal to

$$\pi\mathbb{E}\left[\frac{A(G_h - G_g) + \eta_g\left[G_h^2 - G_g^2 + B(G_h - G_g)\right]}{(G_g + B)(G_h + B)}\right] \tag{310}$$

$$=\pi\mathbb{E}\left[\frac{(G_h - G_g)(A + \eta_g B) + \eta_g(G_h^2 - G_g^2)}{(G_g + B)(G_h + B)}\right], \tag{311}$$

which is positive if $G_h > G_g$. Thus, $\Delta DL_g - \Delta DL_h > 0$ if $G_h > G_g$.

(c) If $G_g = G_h$, then Expression (308) is equal to

$$\pi\mathbb{E}\left[\frac{G_h^2(\eta_h - \eta_g) + BG_h(\eta_h - \eta_g)}{(G_h + B)(G_h + B)}\right] = \pi(\eta_h - \eta_g)G_h\mathbb{E}\left[\frac{G_h + B}{(G_h + B)^2}\right], \tag{312}$$

which is positive if, and only if, $\eta_h > \eta_g$. Thus, $\Delta DL_g - \Delta DL_h > 0$ if, and only if, $\eta_h > \eta_g$.

(d) In a core-periphery network as in Assumption 2 with $h \in \mathcal{N}_{core}$ and $g \in \mathcal{N}_{per}$, it is $G_h = \frac{N-3}{3} + 2G_{per}$, $G_g = G_{per}$, $\eta_h = 0$, and $\eta_g = 1$, and, thus, Expression (308) is equal to

$$\pi\mathbb{E}\left[\frac{A(\frac{N-3}{3} + 2G_{per} - G_{per}) - G_{per}^2 - BG_{per}}{(G_{per} + B)(\frac{N-3}{3} + 2G_{per} + B)}\right] = \pi\mathbb{E}\left[\frac{A\frac{N-3+3G_{per}}{3} - (G_{per} + B)G_{per}}{(G_{per} + B)(\frac{N-3+6G_{per}}{3} + B)}\right]. \tag{313}$$

Moreover, it is

$$A = \sum_{j=1, j\notin\{g,h\}}^{N} D_j G_j \eta_j = G_{per} \sum_{j\in\mathcal{N}_{per}\setminus\{g\}} D_j \tag{314}$$

$$B = \sum_{j=1, j\notin\{g,h\}}^{N} (1 - D_j)G_j = \frac{N - 3 + 6G_{per}}{3} \sum_{j\in\mathcal{N}_{core}\setminus\{h\}} (1 - D_j) + G_{per} \sum_{j\in\mathcal{N}_{per}\setminus\{g\}} (1 - D_j), \tag{315}$$

which implies that the nominator in the expectation in Expression (313) is equal to

$$\tilde{A} = A\frac{N-3+3G_{per}}{3} - (G_{per}+B)G_{per} \tag{316}$$

$$= G_{per}\frac{N-3+3G_{per}}{3}\sum_{j\in\mathcal{N}_{per}\setminus\{g\}} D_j - G_{per}^2 \tag{317}$$

$$- G_{per}\left(\frac{N-3+6G_{per}}{3}\sum_{j\in\mathcal{N}_{core}\setminus\{h\}}(1-D_j) + G_{per}\sum_{j\in\mathcal{N}_{per}\setminus\{g\}}(1-D_j)\right)$$

$$= G_{per}\Bigg(\sum_{j\in\mathcal{N}_{per}\setminus\{g\}}\left(D_j\frac{N-3+3G_{per}}{3} - (1-D_j)G_{per}\right) \tag{318}$$

$$- \frac{N-3+6G_{per}}{3}\sum_{j\in\mathcal{N}_{core}\setminus\{h\}}(1-D_j) - G_{per}\Bigg)$$

$$= G_{per}\left(\sum_{j\in\mathcal{N}_{per}\setminus\{g\}}\left(D_j\frac{N-3+6G_{per}}{3} - G_{per}\right) - \frac{N-3+6G_{per}}{3}\sum_{j\in\mathcal{N}_{core}\setminus\{h\}}(1-D_j) - G_{per}\right) \tag{319}$$

$$= G_{per}\Bigg(\sum_{j\in\mathcal{N}_{per}\setminus\{g\}}\left(D_j\frac{N-3+6G_{per}}{3}\right) - G_{per}\frac{2N-3}{3} - \frac{N-3+6G_{per}}{3}\frac{N-3}{3} \tag{320}$$

$$+ \frac{N-3+6G_{per}}{3}\sum_{j\in\mathcal{N}_{core}\setminus\{h\}} D_j - G_{per}\Bigg)$$

$$= G_{per}\left(\frac{N-3+6G_{per}}{3}\sum_{j\in\{1,\ldots,N\}\setminus\{g,h\}} D_j - G_{per}\frac{2N-3}{3} - \frac{N-3+6G_{per}}{3}\frac{N-3}{3} - G_{per}\right) \tag{321}$$

$$= \hat{a}\hat{D} - \hat{b}, \tag{322}$$

with

$$\hat{a} = G_{per}\frac{N-3+6G_{per}}{3} > 0, \tag{323}$$

$$\hat{b} = G_{per}\left(G_{per}\frac{2N-3}{3} + \frac{N-3+6G_{per}}{3}\frac{N-3}{3} + G_{per}\right) > 0, \tag{324}$$

$$\hat{D} = \sum_{j\in\{1,\ldots,N\}\setminus\{g,h\}} D_j \sim Bin(N-2,\pi). \tag{325}$$

We define $\hat{d} = \hat{b}/\hat{a} > 0$. Then,

$$\tilde{A} \geq 0 \;\Leftrightarrow\; \hat{a}\hat{D} - \hat{b} \geq 0 \;\Leftrightarrow\; \hat{D} \geq \hat{d}. \tag{326}$$

We consider the following two cases:
$\underline{\hat{D} \geq \hat{d}}$: In this case, $\tilde{A} \geq 0$. Then, using that $B \geq 0$, it is

$$\frac{\tilde{A}}{(G_{per} + B)(\frac{N-3+6G_{per}}{3} + B)} \leq \frac{\tilde{A}}{G_{per}\frac{N-3+6G_{per}}{3}}. \tag{327}$$

$\underline{\hat{D} < \hat{d}}$: In this case, $\tilde{A} < 0$. Then, using that

$$B \leq \frac{N-3+6G_{per}}{3}(|\mathcal{N}_{core}| - 1) + G_{per}(|\mathcal{N}_{per}| - 1) = \overline{b} > 0, \tag{328}$$

it is

$$\frac{\tilde{A}}{(G_{per} + B)(\frac{N-3+6G_{per}}{3} + B)} \leq \frac{\tilde{A}}{(G_{per} + \overline{b})(\frac{N-3+6G_{per}}{3} + \overline{b})}. \tag{329}$$

Combining both cases, Expression (313) is equal to

$$\pi\mathbb{E}\left[\frac{A\frac{N-3+3G_{per}}{3}-(G_{per}+B)G_{per}}{(G_{per}+B)(\frac{N-3+6G_{per}}{3}+B)}\right] \tag{330}$$

$$=\pi\Bigg(\mathbb{P}(\tilde{D}\geq\hat{d})\mathbb{E}\left[\frac{\tilde{A}}{(G_{per}+B)(\frac{N-3+6G_{per}}{3}+B)}\mid\tilde{D}\geq\hat{d}\right]$$
$$+\mathbb{P}(\tilde{D}<\hat{d})\mathbb{E}\left[\frac{\tilde{A}}{(G_{per}+B)(\frac{N-3+6G_{per}}{3}+B)}\mid\tilde{D}<\hat{d}\right]\Bigg) \tag{331}$$

$$\leq\pi\Bigg(\mathbb{P}(\tilde{D}\geq\hat{d})\frac{\mathbb{E}\left[\tilde{A}\mid\tilde{D}\geq\hat{d}\right]}{G_{per}\frac{N-3+6G_{per}}{3}}+\mathbb{P}(\tilde{D}<\hat{d})\frac{\mathbb{E}\left[\tilde{A}\mid\tilde{D}<\hat{d}\right]}{(G_{per}+\bar{b})(\frac{N-3+6G_{per}}{3}+\bar{b})}\Bigg) \tag{332}$$

$$=\pi\Bigg(\frac{\mathbb{E}\left[\tilde{A}\right]}{(G_{per}+\bar{b})(\frac{N-3+6G_{per}}{3}+\bar{b})}$$
$$-\mathbb{P}(\tilde{D}\geq\hat{d})\frac{\mathbb{E}\left[\tilde{A}\mid\tilde{D}\geq\hat{d}\right]}{(G_{per}+\bar{b})(\frac{N-3+6G_{per}}{3}+\bar{b})}+\mathbb{P}(\tilde{D}\geq\hat{d})\frac{\mathbb{E}\left[\tilde{A}\mid\tilde{D}\geq\hat{d}\right]}{G_{per}\frac{N-3+6G_{per}}{3}}\Bigg) \tag{333}$$

$$=\pi\Bigg(\frac{\mathbb{E}\left[\tilde{A}\right]}{(G_{per}+\bar{b})(\frac{N-3+6G_{per}}{3}+\bar{b})}$$
$$+\mathbb{P}(\tilde{D}\geq\hat{d})\mathbb{E}\left[\tilde{A}\mid\tilde{D}\geq\hat{d}\right]\left(\frac{1}{G_{per}\frac{N-3+6G_{per}}{3}}-\frac{1}{(G_{per}+\bar{b})(\frac{N-3+6G_{per}}{3}+\bar{b})}\right)\Bigg) \tag{334}$$

$$=\pi\Bigg(\frac{\mathbb{E}\left[\tilde{A}\right]}{(G_{per}+\bar{b})(\frac{N-3+6G_{per}}{3}+\bar{b})}$$
$$+\mathbb{P}(\tilde{D}\geq\hat{d})\mathbb{E}\left[\tilde{A}\mid\tilde{D}\geq\hat{d}\right]\frac{(G_{per}+\bar{b})(\frac{N-3+6G_{per}}{3}+\bar{b})-G_{per}\frac{N-3+6G_{per}}{3}}{G_{per}\frac{N-3+6G_{per}}{3}(G_{per}+\bar{b})(\frac{N-3+6G_{per}}{3}+\bar{b})}\Bigg) \tag{335}$$

$$=\pi\Bigg(\frac{\tilde{a}(N-2)\pi-\hat{b}}{(G_{per}+\bar{b})(\frac{N-3+6G_{per}}{3}+\bar{b})}+\mathbb{P}(\tilde{D}\geq\hat{d})\mathbb{E}\left[\tilde{A}\mid\tilde{D}\geq\hat{d}\right]\tilde{g}\Bigg) \tag{336}$$

with $\tilde{g}=\frac{(G_{per}+\bar{b})(\frac{N-3+6G_{per}}{3}+\bar{b})-G_{per}\frac{N-3+6G_{per}}{3}}{G_{per}\frac{N-3+6G_{per}}{3}(G_{per}+\bar{b})(\frac{N-3+6G_{per}}{3}+\bar{b})}>0$. Using Markov's inequality (note that $\hat{d}>0$), it is

$$\mathbb{P}(\tilde{D}\geq\hat{d})\leq\frac{\mathbb{E}[\tilde{D}]}{\hat{d}}=\frac{(N-2)\pi}{\hat{d}}. \tag{337}$$

Moreover, it is $\mathbb{E}\left[\tilde{A} \mid \tilde{D} \geq \hat{d}\right] \leq \hat{a}(N-2) - \hat{b}$. Using this in Expression (336) yields that

$$\pi\left(\frac{\hat{a}(N-2)\pi - \hat{b}}{(G_{per}+\bar{b})(\frac{N-3+6G_{per}}{3}+\bar{b})} + \mathbb{P}(\tilde{D} \geq \hat{d})\mathbb{E}\left[\tilde{A} \mid \tilde{D} \geq \hat{d}\right]\tilde{g}\right) \tag{338}$$

$$\leq \pi\Bigg(\underbrace{\frac{\hat{a}(N-2)\pi - \hat{b}}{(G_{per}+\bar{b})(\frac{N-3+6G_{per}}{3}+\bar{b})} + \frac{(N-2)\pi}{\hat{d}}(\hat{a}(N-2)-\hat{b})\tilde{g}}_{=\tilde{C}}\Bigg). \tag{339}$$

When π approaches zero, the term inside the parentheses becomes negative:

$$\tilde{C} \to -\frac{\hat{b}}{(G_{per}+\bar{b})(\frac{N-3+6G_{per}}{3}+\bar{b})} < 0 \quad \text{for} \quad \pi \to 0. \tag{340}$$

Because of continuity, there exists $\hat{\pi} > 0$ such that for all $\pi \in (0, \hat{\pi})$ it holds that $\pi\tilde{C} < 0$. Using Equality (308), for $\pi \in (0, \hat{\pi})$ it is, thus, $\Delta DL_g - \Delta DL_h < 0 \Leftrightarrow \Delta DL_g < \Delta DL_h$.

□

Corollary 3. *Consider a core-periphery network and let $g \in \mathcal{N}_{per}$ and $h \in \mathcal{N}_{core}$. If π is sufficiently small, there exists $\hat{\delta} \in (0,1)$ such that $\Delta DL_g = \Delta DL_h$ for the loss sharing rule $w(\hat{\delta})$ and that $\Delta DL_g > \Delta DL_h$ if, and only if, $\delta < \hat{\delta}$.*

Proof. From Proposition 8, it is $\Delta DL_g > \Delta DL_h$ if loss sharing is based on net risk, that is, when δ approaches zero. From Proposition 10 (d), it is $\Delta DL_g < \Delta DL_h$ if loss sharing is based on gross risk ($\delta = 1$) and π is sufficiently small. From Proposition 4, it is $\frac{\partial \Delta DL_g}{\partial \delta} < 0$ and $\frac{\partial \Delta DL_h}{\partial \delta} > 0$, which implies monotonicity of the differential impact of central clearing in δ, that is,

$$\frac{\partial(\Delta DL_g - \Delta DL_h)}{\partial \delta} < 0.$$

Together with continuity, the statement follows. □

G Proofs for Section 5

Lemma 2 (Optimal fee)**.** *For an optimal clearing rule (F^*, δ^*), defined as the solution to (36) subject to (37) and (38), the optimal fee is equal to*

$$F^* = \pi f(K)\xi(\alpha_{uc}) \min_{i \in \Omega}\left(-\Delta DL_i(\delta^*, \Omega)\right), \tag{341}$$

where $\Delta DL_i(\delta, \Omega)$ is the impact of central clearing on i's expected default losses considering only the set Ω of market participants, analogously to Equation (20),

$$\Delta DL_i(\delta, \Omega) = \frac{\mathbb{E}\left[(1 - D_i) \sum_{j \in \mathcal{N}_i \cap \Omega} DL_{ij}^{K-1} + LSC_i(\delta, \Omega)\right]}{\mathbb{E}\left[(1 - D_i) \sum_{j \in \mathcal{N}_i \cap \Omega} DL_{ij}^{K}\right]} - 1. \tag{342}$$

Proof. The participation constraint (37) is equivalent to

$$(1-\pi)F \sum_{j \in \mathcal{N}_i \cap \Omega} |v_{ij}| \leq (1-\pi)\left(\mathbb{E}\left[\sum_{j \in \mathcal{N}_i} DL_{ij}^K - \sum_{j \in \mathcal{N}_i \cap \Omega} DL_{ij}^{K-1} - \sum_{j \in \mathcal{N}_i \setminus \Omega} DL_{ij}^K\right]\right) - \mathbb{E}[LSC_i(\delta, \Omega)] \tag{343}$$

$$\Leftrightarrow (1-\pi)F \sum_{j \in \mathcal{N}_i \cap \Omega} |v_{ij}| \leq (1-\pi)\left(\mathbb{E}\left[\sum_{j \in \mathcal{N}_i \cap \Omega} DL_{ij}^K - \sum_{j \in \mathcal{N}_i \cap \Omega} DL_{ij}^{K-1}\right]\right) - \mathbb{E}[LSC_i(\delta, \Omega)] \tag{344}$$

$$\Leftrightarrow (1-\pi)FG_i(\Omega) \leq (1-\pi)\left(\mathbb{E}\left[DL_i^K(\Omega) - DL_i^{K-1}(\Omega)\right]\right) - \mathbb{E}[LSC_i(\delta, \Omega)] \tag{345}$$

$$\Leftrightarrow \frac{(1-\pi)FG_i(\Omega)}{(1-\pi)\mathbb{E}[DL_i^K(\Omega)]} \leq -\left(\frac{(1-\pi)\mathbb{E}[DL_i^{K-1}(\Omega)] + \mathbb{E}[LSC_i(\delta, \Omega)]}{(1-\pi)\mathbb{E}[DL_i^K(\Omega)]} - 1\right) \tag{346}$$

$$\Leftrightarrow \frac{FG_i(\Omega)}{\mathbb{E}[DL_i^K(\Omega)]} \leq -\Delta DL_i(\delta, \Omega) \tag{347}$$

$$\Leftrightarrow F \leq -\pi f(K)\xi(\alpha_{uc})\Delta DL_i(\delta, \Omega), \tag{348}$$

where $G_i(\Omega)$, $DL_i^K(\Omega)$, and $\Delta DL_i(\delta, \Omega)$ are the gross position, uncleared default loss, and impact of central clearing on the default losses of entity i considering only the set Ω of market participants.

Because the participation constraint must hold for all $i \in \Omega$, it is

$$F^* \leq \min_{i \in \Omega} -\pi f(K)\xi(\alpha_{uc})\Delta DL_i(\delta^*, \Omega) = \pi f(K)\xi(\alpha_{uc}) \min_{i \in \Omega}(-\Delta DL_i(\delta^*, \Omega)). \tag{349}$$

Since the objective function (36) is increasing in F, the optimal clearing fee maximizes F with respect to the participation constraints, which implies that

$$F^* = \pi f(K)\xi(\alpha_{uc}) \min_{i \in \Omega}(-\Delta DL_i(\delta, \Omega)). \tag{350}$$

□

Proposition 11 (Optimal clearing rule)**.** *Consider a core-periphery network. Assume that π is sufficiently small, such that Corollary 3 applies. Then, the optimal clearing rule is one of the following:*

(A) All entities use central clearing, $\Omega = \{1, ..., N\}$, the loss sharing rule balances the impact of central

clearing across entities, $\delta^* = \hat{\delta}$, and the fee is equal to

$$F_A^* = -\pi\xi(\alpha_{uc})f(K)\Delta DL_1(\Omega). \tag{351}$$

(B) *Only core entities use central clearing, $\Omega = \mathcal{N}_{core}$, the loss sharing rule is indeterminate, and the fee is equal to*

$$F_B^* = \pi\xi(\alpha_{uc})(f(K) - f(K-1)). \tag{352}$$

Proof. Entities only differ in whether they are in the core or periphery of the network, but otherwise face the same participation constraints. Let $g \in \mathcal{N}_{per}$ and $h \in \mathcal{N}_{core}$. Let $\hat{\delta} \in (0,1)$ such that $\Delta DL_g(\hat{\delta}, \{1, ..., N\}) = \Delta DL_h(\hat{\delta}, \{1, ..., N\})$, which exists due to Corollary 3. We rewrite the objective function (36) as

$$\mathcal{O} = \sum_{i \in \Omega} \mathbb{E}\left[(1 - D_i) \sum_{j \in \mathcal{N}_i \cap \Omega} \left|v_{ij}\right| F\right] = (1 - \pi)FG(\Omega), \tag{353}$$

where $G(\Omega) = \sum_{i \in \Omega} \sum_{j \in \mathcal{N}_i \cap \Omega} \left|v_{ij}\right|$ is the total gross volume cleared.

Because each peripheral entity trades only with a core entity, it is not feasible that only peripheral entities use central clearing. Therefore, $\mathcal{N}_{core} \subseteq \Omega$. Thus, there are two possible sets of clearing members Ω:[IA.2]

(A) <u>Assume that $\Omega = \{1, ..., N\}$</u>. In this case, all entities use central clearing. Assume that $\delta^* \leq \hat{\delta}$. Then, using Corollary 3, it is $\Delta DL_h(\delta^*, \Omega) \leq \Delta DL_g(\delta^*, \Omega)$, and, thus, using Lemma 2, the optimal fee is equal to

$$F_A^* = \pi f(K)\xi(\alpha_{uc}) \min_{i \in \Omega}(-\Delta DL_i(\delta^*, \Omega)) = -\pi f(K)\xi(\alpha_{uc})\Delta DL_g(\delta^*, \Omega). \tag{354}$$

From Proposition 4, it is $\frac{\partial \Delta DL_g}{\partial \delta} < 0$, and, thus, for all $\delta^* < \hat{\delta}$,

$$\frac{\partial \mathcal{O}(\delta^*)}{\partial \delta} = (1 - \pi)G(\Omega)\frac{\partial F_A^*}{\partial \delta} = -(1 - \pi)\, G(\Omega)\pi f(K)\xi(\alpha_{uc})\frac{\partial \Delta DL_g(\delta^*, \Omega)}{\partial \delta} > 0. \tag{355}$$

Therefore, $\delta^* < \hat{\delta}$ is not optimal.

[IA.2] Ω is nonempty by the assumption in Footnote 24.

Assume that $\delta^* > \hat{\delta}$. Then, $\Delta DL_h(\delta^*, \Omega) > \Delta DL_g(\delta^*, \Omega)$, and, thus, using Lemma 2 it is

$$F_A^* = \pi f(K)\xi(\alpha_{uc}) \min_{i \in \Omega}(-\Delta DL_i(\delta^*, \Omega)) = -\pi f(K)\xi(\alpha_{uc})\Delta DL_h(\delta^*, \Omega). \tag{356}$$

From Proposition 4, it is $\frac{\partial \Delta DL_h}{\partial \delta} > 0$, and, thus, for all $\delta > \hat{\delta}$,

$$\frac{\partial \mathcal{O}(\delta^*)}{\partial \delta} = (1-\pi)G(\Omega)\frac{\partial F_A^*}{\partial \delta} = -(1-\pi)\pi f(K)\xi(\alpha_{uc})G(\Omega)\frac{\partial \Delta DL_h(\delta^*, \Omega)}{\partial \delta} < 0. \tag{357}$$

Therefore, $\delta > \hat{\delta}$ is not optimal, and $\delta^* = \hat{\delta}$ is a maximum. Thus, $\delta^* = \hat{\delta}$ maximizes the CCP's profit.

(B) Assume that $\Omega = \mathcal{N}_{core}$. In this case, only core entities use central clearing. Because core entities have zero net risk, $\bar{\sigma}_j = 0$ for all $j \in \mathcal{N}_{core}$, using Proposition 4, the expected loss sharing contribution is equal to

$$\mathbb{E}[LSC_i] = (1-\pi)\xi(\alpha_{CCP})w_i(\delta)\mathbb{E}\left[\frac{\sum_{j \in \mathcal{N}_{core}, j \neq i} D_j \bar{\sigma}_j}{w_i(\delta) + \sum_{j \in \mathcal{N}_{core}, j \neq i}(1-D_j)w_j(\delta)}\right] = 0. \tag{358}$$

Therefore, for all $i \in \mathcal{N}_{core}$ the impact of central clearing on the expected default losses is equal to

$$\Delta DL_i(\delta, \mathcal{N}_{core}) = \frac{f(K-1) - f(K)}{f(K)}, \tag{359}$$

independently of the loss sharing rule δ. Therefore, using Lemma 2, the optimal fee is equal to

$$F_B^* = -\pi f(K)\xi(\alpha_{uc})\frac{f(K-1) - f(K)}{f(K)} = \pi\xi(\alpha_{uc})(f(K) - f(K-1)). \tag{360}$$

Assume that the loss sharing rule is $\delta^* \in [0, 1]$. If any peripheral entity $g \in \mathcal{N}_{per}$ joins the CCP, the CCP's expected default losses become strictly positive. Thus,

$$\Delta DL_g(\delta^*, \mathcal{N}_{core} \cup \{g\}) > \frac{f(K-1) - f(K)}{f(K)}. \tag{361}$$

From the proof of Lemma 2, entity g prefers <u>not</u> to use central clearing if, and only if,

$$
\begin{aligned}
& -\pi f(K)\xi(\alpha_{uc})\Delta DL_g(\delta^*, \mathcal{N}_{core} \cup \{g\}) < F_B^* && (362)\\
\Leftrightarrow & -\pi f(K)\xi(\alpha_{uc})\Delta DL_g(\delta^*, \mathcal{N}_{core} \cup \{g\}) < \pi\xi(\alpha_{uc})(f(K) - f(K-1)) && (363)\\
\Leftrightarrow & -\Delta DL_g(\delta^*, \mathcal{N}_{core} \cup \{g\}) < \frac{f(K) - f(K-1)}{f(K)} && (364)\\
\Leftrightarrow & \Delta DL_g(\delta^*, \mathcal{N}_{core} \cup \{g\}) > \frac{f(K-1) - f(K)}{f(K)}. && (365)
\end{aligned}
$$

Therefore, constraint (38) holds for all $g \in \mathcal{N}_{per}$.

□

Proposition 12 (Curtailing clearing participation). *In the setting of Proposition 11, clearing rule (B) strictly dominates (A) if*

$$
(f(K) - f(K-1))\,\xi(\alpha_{uc}) < \max\left\{\frac{2N-3}{4N}, \frac{\hat{\delta}}{2}\right\} f(1)\xi(\alpha_{CCP}). \tag{366}
$$

In this case, it is optimal for the CCP to dissuade peripheral entities from using central clearing. There exist $\hat{K} < \infty$ and $\hat{\alpha}_{uc} < 1$ such that Inequality (366) holds if $K > \hat{K}$ or $\alpha_{uc} > \hat{\alpha}_{uc}$.

Proof. Let $k \in \{1, ..., N\}$. Clearing rule (B) results in a strictly larger fee income to the CCP than (A) if, and only if,

$$
\begin{aligned}
& F_B^* G(\mathcal{N}_{core}) > F_A^* G(\{1, ..., N\}) && (367)\\
\Leftrightarrow\ & \pi\xi(\alpha_{uc})(f(K) - f(K-1))G(\mathcal{N}_{core}) > -\pi\xi(\alpha_{uc})f(K)\Delta DL_k(\hat{\delta}, \{1, .., N\})G(\{1, ..., N\}) && (368)\\
\Leftrightarrow\ & \frac{f(K) - f(K-1)}{f(K)} G(\mathcal{N}_{core}) > -\Delta DL_k(\hat{\delta}, \{1, .., N\})G(\{1, ..., N\}) && (369)\\
\Leftrightarrow\ & \frac{f(K) - f(K-1)}{f(K)} G(\mathcal{N}_{core}) > G(\{1, ..., N\}) \left[\frac{f(K) - f(K-1)}{f(K)} - \frac{w_k(\hat{\delta})f(1)}{G_k f(K)} \frac{\xi(\alpha_{CCP})}{\xi(\alpha_{uc})} \frac{1}{\pi} H\right] && (370)\\
\Leftrightarrow\ & \frac{f(K) - f(K-1)}{f(K)} (G(\mathcal{N}_{core}) - G(\{1, ..., N\})) > -G(\{1, ..., N\}) \frac{w_k(\hat{\delta})f(1)}{G_k f(K)} \frac{\xi(\alpha_{CCP})}{\xi(\alpha_{uc})} \frac{1}{\pi} H && (371)\\
\Leftrightarrow\ & \frac{f(K) - f(K-1)}{f(K)} (G(\{1, ..., N\}) - G(\mathcal{N}_{core})) < G(\{1, ..., N\}) \frac{w_k(\hat{\delta})f(1)}{G_k f(K)} \frac{\xi(\alpha_{CCP})}{\xi(\alpha_{uc})} \frac{1}{\pi} H && (372)
\end{aligned}
$$

where $H = \mathbb{E}\left[\frac{\sum_{j=1, j\neq k}^{N} D_j G_j \eta_j}{w_k(\hat{\delta}) + \sum_{j=1, j\neq k}^{N}(1-D_j)w_j(\hat{\delta})}\right]$. In the following, we use that

$$
w_k(\hat{\delta}) = \hat{\delta}G_k f(1) + (1-\hat{\delta})G_k\eta_k f(1) \leq G_k f(1). \tag{373}
$$

(1) Let $k \in \mathcal{N}_{core}$. Then, using the properties of core-periphery networks,

$$H = \mathbb{E}\left[\frac{\sum_{j=1,j\neq k}^{N} D_j G_j \eta_j}{w_k(\hat{\delta}) + \sum_{j=1,j\neq k}^{N}(1-D_j)w_j(\hat{\delta})}\right] = \mathbb{E}\left[\frac{\sum_{j\in\mathcal{N}_{per}} D_j G_{per}}{w_k(\hat{\delta}) + \sum_{j=1,j\neq k}^{N}(1-D_j)w_j(\hat{\delta})}\right] \tag{374}$$

$$\geq \mathbb{E}\left[\frac{\sum_{j\in\mathcal{N}_{per}} D_j G_{per}}{f(1)\sum_{j=1}^{N} G_j(\{1,...,N\})}\right] = \frac{\frac{2N}{3}\pi G_{per}}{G(\{1,...,N\})f(1)}. \tag{375}$$

Because $k \in \mathcal{N}_{core}$, it is $w_k(\hat{\delta}) = \hat{\delta} G_{core} f(1)$. Therefore, Inequality (372) holds if

$$\frac{f(K)-f(K-1)}{f(K)}(G(\{1,...,N\}) - G(\mathcal{N}_{core})) < \frac{\hat{\delta} G_{core} f(1) f(1)}{G_{core} f(K)} \frac{\xi(\alpha_{CCP})}{\xi(\alpha_{uc})} \frac{1}{\pi} \frac{\frac{2N}{3}\pi G_{per} G(\{1,...,N\})}{G(\{1,...,N\})f(1)} \tag{376}$$

$$\Leftrightarrow \frac{f(K)-f(K-1)}{f(K)}(G(\{1,...,N\}) - G(\mathcal{N}_{core})) < \frac{\hat{\delta} f(1)}{f(K)} \frac{\xi(\alpha_{CCP})}{\xi(\alpha_{uc})} \frac{2N}{3} G_{per} \tag{377}$$

$$\Leftrightarrow \frac{f(K)-f(K-1)}{f(K)}\left(\frac{2N}{3}G_{per} + \frac{N}{3}\frac{N-3+6G_{per}}{3} - \frac{N}{3}\frac{N-3}{3}\right) < \frac{\hat{\delta} f(1)}{f(K)} \frac{\xi(\alpha_{CCP})}{\xi(\alpha_{uc})} \frac{2N}{3} G_{per} \tag{378}$$

$$\Leftrightarrow \frac{f(K)-f(K-1)}{f(K)} 4G_{per} < \frac{\hat{\delta} f(1)}{f(K)} \frac{\xi(\alpha_{CCP})}{\xi(\alpha_{uc})} 2G_{per} \tag{379}$$

$$\Leftrightarrow 2\frac{f(K)-f(K-1)}{f(K)} < \hat{\delta}\frac{f(1)}{f(K)} \frac{\xi(\alpha_{CCP})}{\xi(\alpha_{uc})} \tag{380}$$

$$\Leftrightarrow \frac{f(K)-f(K-1)}{f(1)} \frac{\xi(\alpha_{uc})}{\xi(\alpha_{CCP})} < \frac{\hat{\delta}}{2}. \tag{381}$$

(2) Let $k \in \mathcal{N}_{per}$. Then,

$$H = \mathbb{E}\left[\frac{\sum_{j=1,j\neq k}^{N} D_j G_j \eta_j}{w_k(\hat{\delta}) + \sum_{j=1,j\neq k}^{N}(1-D_j)w_j(\hat{\delta})}\right] = \mathbb{E}\left[\frac{\sum_{j\in\mathcal{N}_{per}\setminus\{k\}} D_j G_{per}}{w_k(\hat{\delta}) + \sum_{j=1,j\neq k}^{N}(1-D_j)w_k(\hat{\delta})}\right] \tag{382}$$

$$\geq \mathbb{E}\left[\frac{\sum_{j\in\mathcal{N}_{per}\setminus\{k\}} D_j G_{per}}{f(1)\sum_{j=1}^{N} G_j(\{1,...,N\}}\right] = \frac{\frac{2N-3}{3}\pi G_{per}}{G(\{1,...,N\})f(1)}. \tag{383}$$

Because $k \in \mathcal{N}_{per}$, it is $w_k(\hat{\delta}) = G_{per} f(1)$. Therefore, it is sufficient for Inequality (372) to hold if

$$\frac{f(K)-f(K-1)}{f(K)}(G(\{1,...,N\})-G(\mathcal{N}_{core})) < \frac{G_{per}f(1)f(1)}{G_{per}f(K)}\frac{\xi(\alpha_{CCP})}{\xi(\alpha_{uc})}\frac{1}{\pi}\frac{\frac{2N-3}{3}\pi G_{per}G(\{1,...,N\})}{G(\{1,...,N\})f(1)} \quad (384)$$

$$\Leftrightarrow \frac{f(K)-f(K-1)}{f(K)}(G(\{1,...,N\})-G(\mathcal{N}_{core})) < \frac{f(1)}{f(K)}\frac{\xi(\alpha_{CCP})}{\xi(\alpha_{uc})}\frac{2N-3}{3}G_{per} \quad (385)$$

$$\Leftrightarrow \frac{f(K)-f(K-1)}{f(K)}\frac{N}{3}\frac{12G_{per}}{3} < \frac{f(1)}{f(K)}\frac{\xi(\alpha_{CCP})}{\xi(\alpha_{uc})}\frac{2N-3}{3}G_{per} \quad (386)$$

$$\Leftrightarrow \frac{f(K)-f(K-1)}{f(1)}\frac{4N}{2N-3} < \frac{\xi(\alpha_{CCP})}{\xi(\alpha_{uc})} \quad (387)$$

$$\Leftrightarrow \frac{f(K)-f(K-1)}{f(1)}\frac{\xi(\alpha_{uc})}{\xi(\alpha_{CCP})} < \frac{2N-3}{4N}. \quad (388)$$

Therefore, the CCP strictly prefers rule (B) over (A) if

$$(f(K)-f(K-1))\,\xi(\alpha_{uc}) < \max\left\{\frac{2N-3}{4N}, \frac{\hat{\delta}}{2}\right\} f(1)\xi(\alpha_{CCP}). \quad (389)$$

The LHS converges to zero for $K \to \infty$ (using Lemma IA.2) and for $\alpha_{uc} \to 1$ (using Lemma IA.1). Therefore, there exist $\hat{K} < \infty$ and $\hat{\alpha}_{uc} < 1$ such that the CCP strictly prefers rule (B) over (A) if either $K > \hat{K}$ or $\alpha_{uc} > \hat{\alpha}_{uc}$ or both. □

Proposition 13 (Robust optimal clearing rule). *If clearing rule (B) in Proposition 11 is strictly preferred over (A), then only net-based loss sharing is robust to small perturbations in the following sense:*

There exists a sequence $(n_\ell)_{\ell\in\mathbb{N}}$ that converges to 0 and associates with the following sequence of core-periphery networks:

- *Each peripheral entity has the perturbed position $\tilde{G}^\ell_{per} = G_{per} + n_\ell$.*
- *Peripheral entities always centrally clear n_ℓ, independently of the clearing rule, and centrally clear G_{per} if, and only if, the participation constraint is satisfied.*
- *Core entities use central clearing if, and only if, the participation constraint is satisfied.*

$(F^{*,\ell}, \delta^{*,\ell})$ denotes an optimal clearing rule for the ℓth perturbation. Then, (F^*, δ^*) is a robust optimal clearing rule for the original core-periphery network if $F^{*,\ell} \to F^*$ and $\delta^{*,\ell} \to \delta^*$ for $\ell \to \infty$.

Proof. Consider clearing rule (B) associated with clearing members $\Omega = \mathcal{N}_{core}$ and fee F^*_B. The constraint (38) implies for the original network that peripheral entities strictly prefer not to become clearing members. By continuity, there exists $\bar{\ell} > 0$ such that constraint (38) holds for all perturbed networks with $\ell < \bar{\ell}$.

Let $\ell < \bar{\ell}$ and consider the ℓ-th perturbed network. Note that peripheral entities centrally clear n_ℓ, but not G_{per}. Lemma 2 implies that the optimal fee is

$$F^{*,\ell} = -\pi f(K)\xi(\alpha_{uc})\Delta DL_h(\delta^{*,\ell}, \Omega^{*,\ell}),$$

where $h \in \mathcal{N}_{core}$. Proposition 4 (b) implies that the impact of central clearing on a core entity's expected default loss, ΔDL_h, is increasing with δ. Because the CCP's profit is increasing with the fee $F^{*,\ell}$, it is optimal to maximize $F^{*,\ell}$ by minimizing δ. Thus, $\delta^{*,\ell} = 0$ and, using Proposition 8,

$$\Delta DL_h(0, \Omega^{*,\ell}) = \frac{f(K-1)}{f(K)} + \pi^{2N/3-1} \frac{6n_\ell}{(N-3)+6n_\ell} \frac{1-\pi^{N/3}}{1-\pi} \frac{\xi(\alpha_{CCP})}{\xi(\alpha_{uc})} \frac{f(1)}{f(K)} - 1. \tag{390}$$

Therefore,

$$\lim_{\ell \to \infty} F^{*,\ell} = -\pi(1-\pi)\xi(\alpha_{uc})f(K)\left[\frac{f(K-1)}{f(K)} - 1\right] = F_B^*.$$

Therefore, $(F_B^*, 0)$ is the robust optimal clearing rule for the original core-periphery network.

□